§ Recap of Last Week

1) $\mathcal{W}_M(R) := \mathrm{Hom}_{R}\left(\mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times; R\right)$ — allows "bad" places

If $M = 1$, $\mathcal{W} := \mathcal{W}_1$

This is a rigid analytic space &
$\mathcal{W}_M(R) \underset{/\mathbb{C}_p}{\simeq} (\mathbb{Z}/pM\mathbb{Z})^{*,\vee}(R) \times B(1,1)(R)$
$\hookrightarrow \phi(pM)$-copies of $B(1,1)$

2) $x : \mathbb{Z}_p^\times \to R^\times$ in $\mathcal{W}(R)$ is in $\mathcal{W}[r](R)$ (for some $r>0$) $\rightsquigarrow r(x)$

3) $\mathbb{Z} \subset \mathcal{W}$ (& very Z-dense)

4) $\exists$ analytic function $\log_p^{[k]}(\sigma)$, $\sigma \in \mathcal{W}(\mathbb{C}_p)$

5) $\Lambda := \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \hookrightarrow \mathcal{O}(\mathcal{W})$ (via eval.)

6) $S_0(p) \subset M_2(\mathbb{Z})$ $\quad (S_0(p) \cap SL_2(\mathbb{Z}) = \Gamma_0(p))$

Given $x \in \mathcal{W}(R) \rightsquigarrow$ define wgt x action of $S_0(p)$ on $\mathcal{A}_x^{(+)}(R)$ & $\mathcal{D}_x^{(+)}(R)$

7) SES $\rightsquigarrow U_p$ is cpct on $\mathrm{Symb}_p(\mathcal{D}_x[r](R))$
$\rightsquigarrow (\ )^{\le \nu}$ makes sense

§ Fund'l SES for O.C. Mod. Symb.

Prop [Pollack–Stevens]

Let $k>0$, $L/\mathbb{Q}_p$ finite, $r>0$. Then, there exists a Hecke SES

$$0 \to S_p(\mathcal{D}^{+}_{-2-k}[r](L))(k+1) \to S_p(\mathcal{D}^{+}_k[r](L))$$
$$\xrightarrow{\rho_k} S_p(V_k[r](L)) \to 0$$

where $(k+1) \equiv$ twist by $(\det)^{k+1}$ & ρ_k is res.

Moreover, if $r<p$ & $\nu>0$, then

$$0 \to (\ )^{\leq \nu} \to (\ )^{\leq \nu} \to (S_p(V_k(L)))^{\leq \nu} \to 0$$

as well (& $<\nu$ too).

Proof For the 1^{st} one, use LES of coh. w/ compact supp.

$$0 \to S_p(\mathcal{D}^{+}_{-2-k}[r](L))(k+1) \to S_p(\mathcal{D}^{+}_k[r](L))$$
$$\to S_p(V_k[r](L)) \to H^2_c(\Gamma, \mathcal{D}^{+}_{-2-k}(L)(k+1))$$

& last term $\overset{P.D}{\simeq} H_0(\Gamma, \mathcal{D}^{+}_{-2-k}[r](L)) \underset{\substack{\uparrow \\ \text{last time}}}{=} 0$

For 2^{nd} one, we know $(\ )^{\leq \nu}$ is left-exact.

Prove "$\to 0$" part by hand using 1^{st} one. $\square$

§ Control THM

<u>THM</u> [Stevens] $L/\mathbb{Q}_p$ finite

(i) $\rho_k : S_\pi(D_k^+(L))^{<k+1} \xrightarrow{\sim} S_p(\mathcal{V}_k(L))^{<k+1}$

(ii) Given norm $\|\cdot\|_s$, ρ_k is isometry

<u>Proof</u> $\mathrm{Symb}_\pi(D_{-2-k}(L)(k+1))^{<k+1}$

$= \mathrm{Symb}_p(D_{-2-k}(L))^{<0} = 0$

Also, given $\phi \in$ RHS, can construct a unique $\Phi \in$ LHS s.t. $\rho_k(\Phi) = \phi$ and $\|\Phi\|_2 = \|\phi\|_1$ (method of Višik) $\square$

§ Mellin Transform

Recall that given $g: \mathbb{R}_+^\times \to \mathbb{C}$ w/ rapid decay near 0 & ∞, we have

$$s \mapsto M(g)(s) = \int_0^\infty g(y) y^s \frac{dy}{y}$$

Rmk 1) $L(f,s) = \frac{\Gamma(s)}{(2\pi)^s} M(f(iy))(s)$

2) g is essentially $\int_{\mathbb{R}_+^\times} g(y)(-)\frac{dy}{y}$

& $y \mapsto y^s$ is a char of $\mathbb{R}_+^\times$

3) $M(g)$ is analytic function on the space of characters $\mathbb{R}_+^\times \to \mathbb{C}^\times$

Now, let $\mathcal{R} = \mathbb{Q}_p$-v.sp of analytic fcts on $\mathcal{W}$ (a Fréchet space)

Given $\mu \in \mathcal{D}^\dagger(\mathbb{Q}_p)$, we want $M(\mu) \in \mathcal{R}$

Easy: $\sigma \in \mathcal{W}(R)$ belongs to some $\mathcal{A}[r](R)$

$\mu \in \mathcal{D}^\dagger(\mathbb{Q}_p) \xrightarrow{\text{res}} \mathcal{D}[r](\mathbb{Q}_p) \rightsquigarrow \mu \otimes 1 \in \mathcal{D}[r](R)$

$\rightsquigarrow M(\mu)(\sigma) = \int_{\mathbb{Z}_p^\times} \sigma(z) d\mu(z)$

$= (\mu \otimes 1)(\sigma)$

Def $M_R : \mathcal{D}^\dagger(R) = \mathcal{D}^\dagger(\mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} R \to \mathcal{R} \hat{\otimes}_{\mathbb{Q}_p} R$

Rmk $\mathcal{R} \hat{\otimes}_{\mathbb{Q}_p} R$ is space of fct's on $\mathcal{W} \times \mathrm{Sp}\, R$

that are analytic in 1^{st} var & locally analytic in 2^{nd} var.

These will be the 2 variables for the input of our p-adic L-fct.

Def Let $U \subset \mathbb{Z}_p$ open & define

$$D_U^+(\mathbb{Q}_p) = \{\mu \in D^+(\mathbb{Q}_p) \mid \mu|_U = \mu\}$$

where $\mu|_U(f) = \mu(f \cdot 1_U)$.

Def Let h be a fct on W = union of $p-1$ balls $B(1,1)$.

We say $\mathrm{ord}(h) \leq \nu$ if on each ball, h can be written as

$$h(x) = \sum_{n \geq 0} a_n (x-1)^n \quad w/ \ |a_n| = O(n^\nu)$$

THM $\mu \mapsto M(\mu)$ is an $\cong$ of Fréchet sp b/w

$$D_{\mathbb{Z}_p^\times}^+(\mathbb{Q}_p) \xrightarrow{\sim} \mathcal{R}$$

Also, $\mathrm{ord}(\mu) \leq \nu \iff \mathrm{ord}(M(\mu)) \leq \nu$

Prop Suppose $\mu' = \underbrace{\Theta_{k+1}}_{\text{"differentiation"}}(\mu)$, then

$$M(\mu')(\sigma) = \log_p^{[k+1]}(\sigma) M(\mu)\left(\frac{\sigma}{z^{k+1}}\right)$$

Proof Suffices to do $k=0$. $\square$

§ Applications for non-crit. slopes

Let $f(z) = \sum_{n\geq 1} a_n q^n$, cusp form of wgt $k+2$, level $\Gamma_1(N)$ & nebentype ε.

Let $\mathcal{H} = \langle T_\ell \; (\ell \nmid N), U_\ell \; (\ell \mid N), \langle a \rangle \rangle$

We care about U_p-slope, so need to have $p \mid N$.

If not, f of level $\Gamma_1(N) \rightsquigarrow \Gamma_1(N) \cap \Gamma_0(p)$

Issue: T_p-eigenform $\not\rightsquigarrow U_p$-eigenform

<u>Trick</u> Let α & β be the roots of

$$X^2 - a_p X + p^{k+1} \varepsilon(p)$$

$\rightsquigarrow$ $f_\alpha(z) := f(z) - \beta f(pz)$ and $f_\beta(z) := f(z) - \alpha f(pz)$ both U_p-eigenf.

w/ eigenvalues α & β resp.

We have $\alpha, \beta \in \mathbb{C}$ actually integral, so $\alpha, \beta \in \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and

$$v_p(\alpha), v_p(\beta) \geq 0, \quad v_p(\alpha) + v_p(\beta) = k+1$$

and $v_p(a_p) \geq \min(v_p(\alpha), v_p(\beta))$

w/ $=$ if $v_p(\alpha) \neq v_p(\beta)$.

We always assume $v_p(\alpha) \leq v_p(\beta)$

If $v_p(\alpha) = 0 \rightsquigarrow v_p(a_p) = 0$ (p-ord.)

Then, we say f_α is the ordinary refinement & f_β is the crit. slope "

Using refinements if necessary,
we work w/ a normalized cuspidal eigenform g of weight $k+2$, level $\Gamma \supset \Gamma_0(p)$, nebentype ε & $U_p g = \alpha g$ w/ $v_p(\alpha) < k+1$.

Step 1: $g \rightsquigarrow \dfrac{\phi_g^+}{\Omega_g^+}$ & $\dfrac{\phi_g^-}{\Omega_g^-}$ (Ch. 5)

both in $\mathrm{Symb}_p(V_k(L))$,

where L = finite ext^n of $\mathbb{Q}_p$ gen'd by $K_p = \mathbb{Q}((a_n(g))_n)$

$$\left(\begin{array}{l} \exists \text{ Hecke lin. map } S_{k+2}(\Gamma, \mathbb{C}) \leftrightarrow \mathrm{Symb}_p^{\pm}(V_k(\mathbb{C})) \\ \qquad\qquad f \mapsto \phi_f^{\pm} \\ \text{Essentially, } f \mapsto \left(D \mapsto \left(P \mapsto \int_D f(z)P(z)\,dz\right)\right) \end{array}\right)$$

Step 2: $U_p(\phi_g^{\pm}) = \alpha \phi_g^{\pm}$, so

$$\frac{\phi_g^{\pm}}{\Omega_g^{\pm}} \in \mathrm{Symb}_p^{\pm}(V_k(L))^{<k+1}$$

lifts to unique $\Phi_g^{\pm} \in \mathrm{Symb}_p^{\pm}(\mathcal{D}_k^{+}(L))$

Step 3: $\mu_g^{\pm} := \Phi_g^{\pm}(\{\infty\} - \{0\}) \in \mathcal{D}_k^{+}(L)$

Step 4: $M_g^{\pm} := M(\mu_g^{\pm})$ $\left(\begin{array}{l}\text{CHECK: } M_g^{\pm}(\sigma) = 0 \\ \qquad \text{if } \sigma(-1) \neq \pm 1\end{array}\right)$

Def $L_p^+(g, \sigma) = M_g^+(\sigma)$, $\forall \sigma \in \mathcal{W}^+(\mathbb{C}_p)$

$L_p^-(g, \sigma) = M_g^-(\sigma)$, $\forall \sigma \in \mathcal{W}^-(\mathbb{C}_p)$

$(-1)^j = \pm 1$

1) $\mu_g^{\pm}(z^j) = \dfrac{\Gamma(j+1)}{(2\pi)^{j+1}\Omega_g^{\pm}} L(f, j+1)$ or 0

for $0 \leq j \leq k$

2) More gen'lly, if $\chi(-1)(-1)^j = \pm 1$ & χ has cond p^n, then

$$L_p^{\pm}(g, \chi z^j) = \left(\frac{p^n}{-2\pi i}\right)^{j+1} \cdot \frac{j!}{\alpha^n \tau(\chi^{-1})} \cdot \frac{L(g, \chi^{-1}, j+1)}{\Omega_g^{\pm}}$$

$\uparrow$
$\chi \neq 1$

<u>THM</u> For g as above & a choice of sign $\pm$, $\exists!\ L_p^{\pm}(g, -)$ on W s.t.

1) $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ of cond p^n, $0 \leq j \leq k$ s.t. $\chi(-1)(-1)^j = \pm 1$, then

$$L_p^{\pm}(g, \chi z^j) = e_p(g, \chi, j) \cdot (*) \cdot L(g, \chi^{-1}, j+1)$$

w/ $e_p(g, \chi, j) = 1 \Leftrightarrow \chi \neq 1$

2) $\mathrm{ord}\,(L_p^{\pm}(g, -)) \leq v_p(\alpha)$

(Can formulate this in terms of f and f_α too)