§3 Adjoint p-adic L-function

As usual $p \nmid N$, $\Gamma = \Gamma_0(p) \cap \Gamma_1(N)$

$\mathcal{H} = \langle T_\ell \ (\ell \nmid Np),\ U_p,\ \langle a \rangle \ (a \in (\mathbb{Z}/N\mathbb{Z})^\times) \rangle$

—— o ——

Let R be any Noetherian ring and
T be any finite algebra.

Def $I = \ker(m : T \otimes_R T \to T) \subset T \otimes_R T$
$= \langle t \otimes 1 - 1 \otimes t \mid t \in T \rangle$

Def Given a $T \otimes T$-module M, let
$$M[I] = \{m \in M \mid Im = 0\}$$

Note: $M[I]$ has 2 obvious T-mod struct.
They are the same.

Now, let M, N = T-module which are finite flat $/R$

Let $b : M \times N \to R$ be an R-lin. pairing which is T-equivariant.

Def $L_b := b_T((M \otimes T)[I], (N \otimes T)[I]) \subset T$

$\uparrow$ T-lin. ext'n of b.

<u>Prop</u> Assume $M \simeq N \simeq T^\vee$ as T-modules, then L_b is principal.

More precisely, $b : T^\vee \otimes T^\vee \to R \iff \tilde{b} \in T \otimes T$

$\Rightarrow L_b = m(\tilde{b})\, T$

<u>Proof</u> $(M \otimes T)[I] \simeq (T^\vee \otimes T)[I] = \mathrm{Hom}_T(T, T) = T$

Let g = gen of this module, h = same for $(N \otimes T)[I]$

Then, L_b is gen'd by $b_T(g, h)$

One can check directly that

$$b_T(g, h) = m(\tilde{b})$$

$$b_T(\textstyle\sum e_i^\vee \otimes e_i, \sum e_i^\vee \otimes e_i) \qquad m\big(\sum_{i,j} b(e_i^\vee, e_j^\vee)\, e_i \otimes e_j\big)$$

$\square$

—— o ——

We will use this idea to construct an "L-adjoint" ideal sheaf on $\mathcal{E}^\circ$.

We work with $W = \mathrm{Sp}\, R \hookrightarrow \mathcal{W}$ which belongs to the covering $\mathcal{C}$ (so R & its residue ring $\tilde{R}$ are PID & $\mathbb{N} \cap W$ is very Zar-dense in W)

Let $K : \mathbb{Z}_p^\times \to R^\times \iff W \in \mathcal{W}(R)$

Let $\nu \geq 0$ adapted to W & $0 < r < \max(r(W), p)$

We will construct an R-bilin. pairing b on $\mathrm{Symb}_p(\mathcal{D}_K[r](R))^{\leq \nu}$ that interpolates a "classical" pairing on $\mathrm{Symb}_p(V_k[r](\mathbb{Q}_p))^{\leq \nu}$ for $k \in \mathbb{N} \cap W(\mathbb{Q}_p)$.

§3 Bilinear Pairing in Weight k

Fix $k \geq 0$, $L/\mathbb{Q}_p$ finite.

We have
$$\rho_k : \mathcal{D}_k[1](L) \to V_k(L)$$
$$i_k : P_k(L) \to A_k[1](L)$$
dual

Def Given $\Phi_1, \Phi_2 \in \mathrm{Symb}_p(\mathcal{D}_k[1](L))$, let
$$[\Phi_1, \Phi_2]_k := [\rho_k(\Phi_1), \rho_k(\Phi_2)]$$
where $[\cdot, \cdot] : \mathrm{Symb}_p(V_k(L)) \otimes \mathrm{Symb}_p(V_k(L)) \to L$ is defined on classical mod. symbols in Chap 5.

Rmk We can restrict $[\cdot, \cdot]_k$ to
$$\mathrm{Symb}_p(\mathcal{D}_k[1](L))^{\leq \nu} = \mathrm{Symb}_p(\mathcal{D}_k[r](L))^{\leq \nu}$$
where $0 < r < p$.

Lemma The operators in $\mathcal{H}$ are self-adjoint w.r.t. $[\cdot,\cdot]_k$.

Proof ρ_k is $\mathcal{H}$-equivariant & this is true for the classical pairing. $\square$

Def Let $\Delta : \mathcal{D}_k[r](L) \times \mathcal{H}_k[r](L) \to L$ be the natural pairing. It extends to a pairing on $\mathrm{Symb}_p(\cdots)$.

Def Let $\Phi \in \mathrm{Symb}_p(\mathcal{D}_k[r](L))$

We define $V_k(\Phi) \in \mathrm{Symb}_p(\mathcal{H}_k[r](L))$ as

$$V_k(\Phi)(D) = (-1)^k \sum_{i \geq 0} (N_p)^i \binom{k}{i} \times \left(\Phi(W_{N_p} \cdot D)\left(z^i(1 - N_p e z)^{k-i}\right)\right) \times (z-e)^i$$

This is its Taylor series for any $z, e \in \mathbb{Z}_p$ s.t. $|z-e| < r$.

Prop $[\Phi_1, \Phi_2]_k = \Delta(\Phi_1, V_k(\Phi_2))$

Def Let $i \in \mathbb{N}$, then

(i) Given $y \in \mathbb{R}$, $\binom{y}{i} := \begin{cases} \frac{y(y-1)\cdots(y-i+1)}{i!}, & i>0 \\ 1, & i=0 \end{cases}$

(ii) Given $x \in W$, $\binom{x}{i} = \binom{\log_p(x(\gamma))/\log_p(\gamma)}{i}$,

where γ is a generator of $1+p\mathbb{Z}_p$

Remark The formula for V_k makes sense w/ $k \rightsquigarrow x$

Def Given Φ, D, x, e & $r < r(x)$,

$$T_{x,\Phi,D,e}(z) := x(-1) \sum (N_p)^i \binom{x}{i} \times \Phi(W_{N_p} \cdot D)\left(z^i (1-N_p e z)^{-i} x(1-N_p e z)\right) \times (z-e)^i$$

Lemma If $r < \min(r(x), p^{-1/p-1}) =: m(x,p)$, then the series above converges on the closed ball $|z-e| \leq r$.

Prop $x \in W$, $r < m(x,p)$. Then,

(i) $T_{x,\Phi,D_e}(z) \in \mathcal{H}_x[r](R)$

(ii) Fix $\Phi \in \mathrm{Symb}_p(\mathcal{D}_x[r](L))$.

Define $V_x(\Phi) \in \mathrm{Hom}(\Delta_0, \mathcal{H}_x[r](L))$ by sending $D \in \Delta_0$ to the element in (i).

Then $V_x(\Phi) \in \mathrm{Symb}_p(\mathcal{H}_x[r](L))$.

Def Let $L = \mathbb{Q}_p$-affinoid algebra,

$x \in \mathcal{W}(L)$, $r < m(x,p)$, $\Phi_i \in S_p(\mathcal{D}_x[r](L))$.

$$[\Phi_1, \Phi_2]_x := \Delta(\Phi_1, V_x(\Phi_2))$$

THM (i) This scalar product comm. w/ affinoid base change $L \mapsto L'$

(ii) When $L = \mathbb{Q}_p$ & $x = k \in \mathbb{N}$, $[\cdot,\cdot]_x = [\cdot,\cdot]_k$

(iii) All $T \in \mathcal{H}$ are self-adj. for $[\cdot,\cdot]_x$

Proof (i) & (ii) triv.

(iii) Let $W = \mathrm{Sp}\, R \subset \mathcal{W}$ admissible in C.

Then, result is true $\forall k \in \mathbb{N} \cap W$. By density, true for (R, K). By (i), true for any (L, x) $\square$

THM Let $\nu \geq 0$ & $L/\mathbb{Q}_p$ finite.
The restriction $[\cdot,\cdot]_\kappa$ to $H^1_!(\Gamma, \mathcal{D}_\kappa[r](L))^{\leq \nu}$ is perfect if $\kappa \notin \mathbb{N}$ or $\kappa = k \in \mathbb{N}$ & $\nu < k+1$.

"Proof" $[\overset{\text{fixed}}{\Phi_1}, \Phi_2] = 0, \ \forall \Phi_2 \in H^1_!(\Gamma, \mathcal{D}_\kappa[r](L))^{\leq \nu}$

$\Rightarrow V_\kappa(\Phi_1) = 0$

$\Rightarrow \forall D \in \Delta_0$ & $i \in \mathbb{N}$, $\binom{\kappa}{i} \Phi_1(D)(z^i) = 0$

If $\kappa \notin \mathbb{N}$, $\binom{\kappa}{i} \neq 0 \Rightarrow \Phi_1 = 0$

If $\kappa = k$ & $\nu < k+1$, $\binom{k}{i} \neq 0$ for $0 \leq i \leq k$

$\Rightarrow \Phi_1(D)(z^i) = 0$ for all D & $0 \leq i \leq k$

$\Rightarrow \rho_k(\Phi_1) = 0 \xRightarrow[\text{ctrl thm}]{\text{Stevens'}} \Phi_1 = 0.$ $\square$

§§ Cohomological Eigenvariety

$$(M_W^\bullet) : 0 \to \overset{-1}{\mathcal{D}_k(R)[r]^{\Gamma})^{\pm}} \to \overset{0}{BS_p^{\pm}(\mathcal{D}_k(R)[r])} \to \overset{1}{S_p(\ldots)} \to 0$$

This is a complex w/ $\neq 0$ coh. only in degree 1:

$$H^1(M_W^\bullet) = (H_!^1)^{\pm}(Y_p, \mathcal{D}_k[r](R))$$

Using section 3.4, we get coh. eigencurves $\mathcal{E}_!^+$ & $\mathcal{E}_!^-$

By construction, the Hecke alg. acting on $H^1(M_W^\bullet)^{\leq \nu}$ is a quotient of the one for $S_p^{\pm}(\mathcal{D}_k[r](R))^{\leq \nu}$ $\Rightarrow$ $\mathcal{E}_!^{\pm}$ is a closed subvariety of $\mathcal{E}^{\pm}$.

<u>Prop</u> $\mathcal{E}_!^+$ & $\mathcal{E}_!^+$ are reduced & $\mathcal{E}_0 = \mathcal{E}_!^+ = \mathcal{E}_!^-$

<u>"Proof"</u> $S^{k+2}(\Gamma, L)^{\leq k+1} = (H_!^1)^+(Y_p, V_k)^{\leq k+1} = (H_!^1)^-(Y_p, V_k)^{\leq k+1}$ from Ch. 5 $\Rightarrow$ $\mathcal{E}_0$, $\mathcal{E}_!^+$ & $\mathcal{E}_!^-$ all share the same classical structure. ☒

<u>Rmk</u> This allows us to work w/ $\mathcal{E}^0$ & overconvergent modular symbols at the same time.

If $x \in \mathcal{C}^0(L)$, $\mathcal{U} = \mathrm{Sp}\,\mathbb{T}$ clean nbgh,
let $W = \mathrm{Sp}\,R = \kappa(\mathcal{U})$, $(K : \mathbb{Z}_p^\times \to R^\times)$

$$\mathcal{M}^\pm = \varepsilon H^1_!(\mathcal{D}_K[r](R))^\pm \quad \text{(indep. of } r>0 \text{ small)}$$

finite free $/R$
& natural $\mathbb{T}$-module

as usual.

There's an obvious analogue of "good points" on $\mathcal{C}^0$:

<u>Lemma</u> $x \in \mathcal{C}^0(L)$, $\omega = \kappa(x) \in \mathcal{W}$, TFAE:

(1) For both $\pm$, $H^1_!(Y_p, \mathcal{D}_\omega(L))^\pm_{(x)}$ is free of rank 1 $/ \mathbb{T}_x$

(2) For both $\pm$ & any clean $\mathcal{U}$, $(\mathcal{M}^\pm)^\vee$ is flat of rank 1 over $\mathbb{T}$ at x

(3) For both $\pm$ & small clean $\mathcal{U}$, $(\mathcal{M}^\pm)^\vee$ is free of rank 1 over $\mathbb{T}$ at x.

Again: $x \in \mathcal{C}^0$ smooth w/ $\kappa(x)(z) = z^k \varepsilon(z)$, & near away from p $\Rightarrow$ x good.

<u>Lemma</u> Suppose $x \in \mathcal{C}^0(L)$ is good.

1) $H^1_!(\mathcal{D}_\infty(L))^{\pm}[x]$ has dim 1

2) If $\kappa(x) = k \in \mathbb{N}$,

$$\dim H^1_!(\mathcal{D}_k(L))^{\pm}[x] = \begin{cases} 1, & x \text{ is VC cusp.} \\ 0, & \text{o.w.} \end{cases}$$

3) The same is true for $H^1_!(\mathcal{D}_k(L))^{\pm}_{(x)}$ if $U_p(x)^2 \neq p^{k+1}\langle p\rangle(x)$

4) If $\kappa(x) = k \in \mathbb{N}$,

both 1-dim

$$\rho_k : H^1_!(\mathcal{D}_k(L))^{\pm}[x] \longrightarrow H^1_!(\mathcal{V}_k(L))^{\pm}[x]$$

is an $\cong$ if κ is étale at x (& 0 o.w.)

5) If $\kappa(x) = k \in \mathbb{N}$ & $v_p(U_p(x)) < k+1$, then x is VC cuspidal & κ is étale at x.

§§ Adjoint L-ideal sheaf

<u>Def</u> $x \in \mathcal{C}^\circ(L)$, U clean nbgh,
$$\mathcal{L}_U^{adj} := L\text{-ideal of } [\cdot,\cdot]_K : M^+ \times M^- \to R$$

<u>Rmk</u> · M^+ & M^- are both quotients of the space $\mathrm{Symb}_p(\mathcal{D}_K[r](L))$ on which $[\cdot,\cdot]_K$ is defined. The pairing factors through.

· $\mathcal{L}_U^{adj}$ is an ideal of $\mathcal{T}$, we can glue them ($\forall x$ & U) to get an ideal sheaf $\mathcal{L}^{adj}$ on $\mathcal{C}^\circ$.

<u>THM</u> The closed analytic subspace defined by $\mathcal{L}^{adj}$ is contained in the non-étale locus of κ

<u>Prop</u> If $x \in \mathcal{C}^\circ(L)$, $x \in U$ clean & small, then $\mathcal{L}_U^{adj} \subset \mathcal{T}$ is principal
$$\Rightarrow \mathcal{L}_U^{adj} = (L_U^{adj}) \quad (\text{w.d. up to } \mathcal{T}^\times)$$

Prop $\varkappa$ & $\mathcal{U}$ as above.

For every $r>0$ small, $\exists\, 0<c<C$ s.t. $\forall y \in \mathcal{U}$ with $\varkappa(y)=\omega$,

$$c\left|[\Phi_y^+, \Phi_y^-]_\omega\right| \leq \left|L_p^{adj}(y)\right| \leq C\left|[\Phi_y^+, \Phi_y^-]_\omega\right|$$

where $\Phi_y^{\pm}$ are gen of $H^1_!(\Gamma, \mathcal{D}_\omega[r](L))^{\pm}[y]$ s.t. $\|\Phi_y^{\pm}\|_r = 1$.

THM $x \in \mathcal{E}^0(L)$ good.

1) If $\varkappa(x) \notin \mathbb{N}$ or x is VC cuspidal, then $L^{adj}(x)=0 \iff \varkappa$ is not étale at x

2) O.w. (if $\varkappa(x)\in\mathbb{N}$ & x is not VCC), then $L^{adj}(x)=0$

THM $x \in \mathcal{E}^0(L)$ smooth good, $e(x)=$ ramif. index of $\varkappa$ at x, then

(1) If $\varkappa(x)\notin\mathbb{N}$ or x is VC w/ slope $< k+1$, then $\quad$ $\varkappa(x)\in\mathbb{N}$ $\quad$ $\shortparallel$

$$\mathrm{ord}_x L^{adj}(x) = e(x)-1$$

2) If $\varkappa(x)=k\in\mathbb{N}$ & x is VCC w/ slope $=k+1$,

$$\mathrm{ord}_x L_p^{adj}(x) = 2e(x)-2$$

3) If $\varkappa(x)=k\in\mathbb{N}$ & x is not VCC, $\mathrm{ord}_x L^{adj}(x) \geq 2e(x)-1$

§3 Hida's Formula

<u>Def</u> Let $M \in \mathbb{N}$, $f \in S_{k+2}(\Gamma_1(M), \varepsilon, \mathbb{C})$ be a newform.

Write $f = \sum a_n q^n$, $a_1 = 1$ &

$$X^2 - a_\ell X + \ell^{k+1}\varepsilon(\ell) = (X - \alpha_\ell)(X - \beta_\ell)$$

$$\Rightarrow L^{adj}(f, s, \psi) = \prod_\ell (1 - \psi(\ell)\alpha_\ell^2 \ell^{-s})$$
$$\times (1 - \psi(\ell)\alpha_\ell \beta_\ell \ell^{-s})$$
$$\times (1 - \psi(\ell)\beta_\ell^2 \ell^{-s})$$

<u>THM</u> $\cdot$ $L^{adj}(f, k+2, \varepsilon^{-1}) = \dfrac{2^{2k+4}\pi^{k+3}}{(k+1)!\,\delta(M) M N(\varepsilon)\varphi\left(\frac{M}{N(\varepsilon)}\right)} (f,f)_{\Gamma_1(M)}$

where $\delta(M) = \begin{cases} 2, & M = 1, 2 \\ 1, & M \geq 3 \end{cases}$

& $N(\varepsilon)$ = cond. of ε.

- $[\phi^+_{f\alpha}, \phi^-_{f\alpha}] = A \cdot B \cdot L^{adj}(f, k+2, 1)$ for explicit values of A & B
- If x, U as above & $y \in U \leftrightarrow f_\alpha$ then

$$c|B| \left| \frac{L^{adj}(f, k, 1)}{\pi^{k+3}\Omega_f^+ \Omega_f^-} \right| < \left| L_p^{adj}(y) \right| < C|B| \left| \frac{L^{adj}(f, k, 1)}{\pi^{k+3}\Omega_f^+ \Omega_f^-} \right|$$