Fix $p \nmid N$ & choice of sign $\pm$

$\Gamma = \Gamma_0(p) \cap \Gamma_1(N)$

<u>Lemma</u> $L/\mathbb{Q}_p$, $x \in \mathcal{E}(L)$, $\kappa(x) = \omega \in \mathcal{W}$

TFAE:

(1) $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_\omega^\dagger(L))^\vee_{(x)}$ is free of rk 1 over $\mathbb{T}_x$ = eigenalgebra of fiber of κ at x

(2) For any clean nbgh $\mathcal{U}$ of x, let $M = e_r \, \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w[r])^{\le \nu}$

↳ idempotent associated to $\mathcal{U} \subset \mathcal{E}_{w,\nu}^\pm$

Then, $M^\vee$ is flat of rk 1 over $\mathbb{T}$ at x, where $\mathcal{U} = \mathrm{Sp}\,\mathbb{T}$.

(3) For any "small" clean nbgh $\mathcal{U}$ of x, $M^\vee$ is free of rk 1 over $\mathbb{T}$.

<u>Proof</u> (2) $\Leftrightarrow$ (3) obvious.

(1) $\Leftrightarrow$ (2): Since $\kappa^{-1}(\omega) \cap \mathcal{U} = \{x\}$, then $\mathbb{T}_x = \mathbb{T}_\omega$ = fiber of $\mathbb{T}$ at ω & $M_\omega = \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_\omega^\dagger(L))_{(x)}$

$\Rightarrow M_\omega^\vee = \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_\omega^\dagger(L))^\vee_{(x)}$. Then, use Nakayama + flatness. ∎

Def $x \in \mathcal{E}^{\pm}$ is good if above holds.

Rmk If x is good $\Rightarrow$ $\forall$ small clean nbgh $\mathcal{U}$ of x, all $y \in \mathcal{U}$ are good.

Prop If x is good, $\kappa(x) = \omega$, then
$$\dim_L \underbrace{\mathrm{Symb}_p^{\pm}(\mathcal{D}_\omega^{\dagger}(L))}_{S}[x] = 1$$

Proof $S[\mathfrak{m}_x] = S[x]$, $\mathfrak{m}_x = \mathfrak{m}_{T_x}$, is dual to $S^\vee/\mathfrak{m}_x S^\vee$ & $S^\vee$ is free of rk $1_\square$

Fact If x is classical, normal & good, its minimal tame level is N.

THM Let $x \in \mathcal{E}^{\pm}$, $\omega = \kappa(x) = (z \mapsto z^k \varepsilon(z))$
If x has min. tame lvl N & $\mathcal{E}^{\pm}$ is smooth at x $\Rightarrow$ x is good.
Moreover, let $\mathcal{W}^{\pm} = \{\sigma \in \mathcal{W} \mid \sigma(-1) = \pm\}$
& $\mathcal{O}_{\mathcal{W},\omega}$ = local ring of $\mathcal{W}^{\pm}$ at ω
Let u be a uniformizer of $\mathcal{O}_{\mathcal{W},\omega}$.
$\Rightarrow$ $\mathcal{O}_{\mathcal{E}^{\pm},x} \cong \mathcal{O}_{\mathcal{W},\omega}[t]/(t^e - u)$,
where e = ram. index of κ at x
& $T_x = L[t]/(t^e)$

Proof Choose clean nbhd $x \in \mathcal{U} \subset \mathcal{E}^{\pm}_{W,\omega}$ w/ $W = \mathrm{Sp}\,R$, T & M as above.

By assumption/construction, $\mathcal{O}_{\mathcal{E},x}$ is a PID.
By some exercise, M_x & $M_x^\vee$ are torsion-free $/ \mathcal{O}_{\mathcal{E}^{\pm},x}$ $\Rightarrow$ free.
So, shrinking $\mathcal{U}$ if necessary, we can assume M & $M^\vee$ are free $/ T$.
$\Rightarrow \mathrm{rk}_R M = \mathrm{rk}_R M^\vee =: d$ & $\mathrm{rk}_R T = e$.
Like last week, we can argue $d = e$.
$\Rightarrow \mathrm{rk}_T M^\vee = 1$ $\Rightarrow$ x is good.
The rest follows from $\mathcal{O}_{\mathcal{E}^{\pm},x} / \mathcal{O}_{W,\omega}$ being an extension of DVR's. $\square$

Def Let $x \in \mathcal{E}^{\pm}(L)$ good. Let
$$\Phi_x^{\pm} \in \mathrm{Symb}_p^{\pm}(\mathcal{D}_\omega^{\dagger}(L))[x]$$
be any generator, $\mu_x^{\pm} = \Phi_x^{\pm}(\{\infty\} - \{0\})$
& $L_p^{\pm}(x, \sigma) = \mu_x^{\pm}(\sigma)$, $\sigma \in \mathcal{W}^{\pm}(\mathbb{C}_p)$,
so $L_p^{\pm}(x, -) \in \mathcal{O}(\mathcal{W}^{\pm}(\mathbb{C}_p))$

Prop $L_p^{\mathcal{I}}(x,\sigma)$ is a function of order $\leq v_p(U_p(x))$.

Proof $\mu_x^{\mathcal{I}}$ is a dist of ord $\leq$ slope at x $\square$

Let $x \in \mathcal{W}(L)$, $g \in M_x^{\dagger}(\Gamma, L)$ be an overconvergent $\mathcal{H}$-eigenform.
It defines a point $x \in \mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$.

Def $L_p^{\mathcal{I}}(g, -) := L_p^{\pm}(x, -)$ (for any $\mathcal{I}$ that makes sense) is an analytic fct on half of the weight space.

Prop Let $g \in S_{k+2}(\Gamma, L)^{<k+1}$, $x = x_g \in \mathcal{C}$.
Then, $x \in \mathcal{C}^+$ & $x \in \mathcal{C}^-$ & is good in both.

Proof we already know
$$x \in \mathcal{C}^0 \subset \mathcal{C}^+ \,\&\, \mathcal{C}^-$$

Let $\lambda = \mathcal{H}$-eigensystem of g.
(If $\lambda(U_p)^2 \neq p^{k+1}\lambda(\langle p \rangle)$) then $\mathcal{C}^+$ & $\mathcal{C}^-$ are both étale at $x \Rightarrow x$ is good. $\square$

So we can define both

$$L_p^+(g,\sigma) \;\&\; L_p^-(g,\sigma)$$

(both up to mult. in $L^\times$).

However, we had already defined

$$g \longleftrightarrow \phi_g^\pm \in \mathrm{Symb}_p^\pm(\mathcal{D}_k(L))$$

(require $\phi_g^\pm \in \mathrm{Symb}_p^\pm(V_k(\mathcal{O}_L))$, not in $\mathfrak{m}_L$)
makes $\phi_g^\pm$ well-def up to $\mathcal{O}_L^\times$)

$$\xleftrightarrow[\text{non-critical slope}]{} \Phi_g^\pm \in \mathrm{Symb}_p^\pm(\mathcal{D}_k^\dagger(L))^{<k+1}$$

$$\mu_g^\pm := \Phi_g^\pm(\{\infty\}-\{0\})$$

$$L^\pm(g,\sigma) := \mu_g^\pm(\sigma)$$

__Prop__ Up to mult. in $L^\times$, $L^\pm(g,-)$ & $L_p^\pm(g,-)$ are the same.

__Proof__ Let $x = x_g \in \mathcal{E}^\pm$. Since g has tame level N, $\mathrm{Symb}_p^\pm(V_k(L))[x]$ has dim 1 & so does $\mathrm{Symb}_p^\pm(\mathcal{D}_k^\dagger(L))[x]$. $\square$

<u>THM</u> Let $y = E_{k+2,\tau,\psi,\mathrm{ord}} \in \Sigma_k(\Gamma, L)$. Then, $x \in \mathcal{C}$ belongs to $\mathcal{C}^{\tau(-1)}$ but not $\mathcal{C}^{-\tau(-1)}$. It is good in $\mathcal{C}^{\tau(-1)}$.

The distribution $\mu_y^{\tau(-1)}$ has support contained in $\{0\}$ & $L_p^{\tau(-1)}(y, -) \equiv 0$ on $\mathcal{W}^{\tau(-1)}$.

<u>Proof</u> $\phi_x^{\tau(-1)} \in \mathrm{Symb}_\Gamma^{\tau(-1)}(\mathcal{D}_k(L))$ is ordinary $\Rightarrow$ $= \sum \phi_{k,u,v}$ (u, v rel prime integers, u/v is a p-integer)

where $\phi_{k,u,v}$ is a classical bdy symbol $\Rightarrow$ $\Phi_x^{\tau(-1)} = \sum \Phi_{k,u,v}$

Since $\mathrm{supp}(\Phi_{k,u,v})$ does not contain ∞,

$$\Phi_{k,u,v}(\{\infty\}-\{0\})(f) = -\Phi_{k,u,v}(\{0\})(f)$$
$$= -f(0)$$

$\Rightarrow$ $\mu_x^{\tau(-1)}$ is a mult. of δ_0 $\square$

THM $g \in M_{k+2}(\Gamma, L)$, crit. slope $k+1$, tame lev N, $\mathcal{H}$-eigenform.

Then, $g = f_\beta$ = crit. refinement of some

$$f \in M_{k+2}(\Gamma_1(N), L)$$

Assume f is normal.

Then, $x = x_g \in \mathcal{C}$ is in both $\mathcal{C}^+$ & $\mathcal{C}^-$, smooth and good in both.

e = ram. index of κ at x in $\mathcal{C}^+$

= ——————————— $\mathcal{C}^-$

We can define $L^\pm(y, \sigma)$ on $\mathcal{W}^\pm$, of order $\le k+1$ & satisfying:

For any $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ finite, $\mathrm{cond}(\chi) = p^n$; & $0 \le j \le k$ s.t. $\chi(-1)(-1)^j = \pm$,

$$L_p^\pm(f_\beta, \chi z^j) = \begin{cases} \dfrac{e_p(f, \alpha, \chi, j)\, p^{n(j+1)}\, j!}{\beta^n (-2\pi i)^{j+1} \tau(\chi^{-1}) \Omega_g^\pm} L(f, \chi^{-1}, j+1), & \text{if } e = 1 \\ 0, \text{ o.w.} \end{cases}$$

If f is CM by K, we can say more. & f is cusp

Let $f \in S_1(\Gamma_1(N), \varepsilon, \mathbb{C})$ newform,

$$X^2 + a_p X + \varepsilon(p) = (X-\alpha)(X-\beta)$$

Assume f regular, i.e. $\alpha \neq \beta$.

Then, both f_α & f_β are ord & crit and distinct.

<u>THM</u> $x \in \mathcal{C}^+$ & $\mathcal{C}^-$ & is good in both

$\leadsto L_p^{\pm}(f, \sigma)$ exists for both $+$ & $-$.

<u>THM</u> Let $x \in \mathcal{E}^I$ good

Then, for some small clean nbgh

$$x \in \mathcal{U} = \mathrm{Sp}\, \tilde{T},$$

$\exists\, \tilde{\mu} \in \mathcal{D}^{\dagger}(T)$ (well-def up to $\mathcal{O}(\mathcal{U})^{\times} = T^{\times}$)

s.t. $\forall y \in \mathcal{U}(L)$, $ev_y(\tilde{\mu}) = \mu_y \in \mathcal{D}^{\dagger}(L)$

defined above,

i.e. $\exists\, c(y) \in L^{\times}$ s.t. $ev_y(\tilde{\mu}) = c(y)\mu_y^{\pm}$

<u>Proof</u> Let $M = \varepsilon\, \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{D}_w[\varepsilon](L))^{\pm\vee}$ assoc'd to $\mathcal{U}$. Choose $\mathcal{U}$ small enough so that $M^{\vee}$ is free of rk 1 / T & M is finite free / R.

Choose $M^{\vee} = T.u$ (some $u \in M^{\vee}$). (up to $T^{\times}$)

Let $\tilde{\iota}_u : M^{\vee} \to T$ be the inv of $t \mapsto tu$

$$\mathrm{Hom}_R(M, \mathcal{D}^{\dagger}(R)) \xrightarrow{\sim} M^{\vee} \otimes_R \mathcal{D}^{\dagger}(R)$$

$$\tilde{e} \mapsto \quad \xrightarrow[\sim]{\tilde{\iota}_u} T \otimes \mathcal{D}^{\dagger}(R) \xrightarrow{\sim} \mathcal{D}^{\dagger}(T) \ni \tilde{\mu}$$

Note $\exists$ canonical $\tilde{e} \in \mathrm{Hom}_R(M, \mathcal{D}^{\dagger}(R))$,

$$\tilde{e}(\Phi) := \Phi(\{\infty\} - \{0\})$$

To prove interpolation, consider

$$\begin{tikzcd}
\tilde{\varepsilon}, \mathrm{Hom}_R(M, \mathcal{D}^\dagger(R)) \arrow[r, "\sim"] \arrow[d] & M^\vee \otimes_R \mathcal{D}^\dagger(R) \arrow[r, "\sim"] \arrow[d] & T \otimes \mathcal{D}^\dagger(R) \arrow[r, "\sim"] \arrow[d] & \mathcal{D}^\dagger(T) \ni \tilde{\mu} \arrow[d] \\
\mathrm{Hom}_L(M_\omega, \mathcal{D}^\dagger(L)) \arrow[r, "\sim"] \arrow[d, "\mathrm{res}"] & M^\vee_\omega \otimes_L \mathcal{D}^\dagger(L) \arrow[r, "\sim"] \arrow[d] & T_\omega \otimes \mathcal{D}^\dagger(L) \arrow[r, "\sim"] \arrow[d] & \mathcal{D}^\dagger(T_\omega) \arrow[d] \\
\mathrm{Hom}_L(M_\omega[y], \mathcal{D}^\dagger(L)) \arrow[r, "\sim"] & M^\vee / \mathfrak{p}_y M^\vee L \otimes \mathcal{D}^\dagger(L) \arrow[r, "\sim"] & T/\mathfrak{p}_y \otimes \mathcal{D}^\dagger(L) \arrow[r, "\sim"] & \mathcal{D}^\dagger(L)
\end{tikzcd}$$

"eval at $\{\infty\}-\{0\}$" $\longmapsto u_y \otimes \mu_y \xmapsto[\text{of } \mu_y]{\text{mult}} = ev_y(\tilde{\mu})$

$\{\Phi_y\} \subseteq M_\omega[y]$ is dual bases to $\{u_y\} \subseteq M^\vee / \mathfrak{p}_y M^\vee$

$\Rightarrow$ Done since $\mu_y = \Phi_y(\{\infty\}-\{0\})$

Here $y \in \mathcal{U}$ & $L = T/\mathfrak{p}_y$ $\square$

<u>Cor</u> $\exists$ 2-var analytic fct $\hat{L}_p$ on $\mathcal{U} \times \mathcal{W}$ s.t. $\forall y \in \mathcal{U}$ & $\sigma \in \mathcal{W}$,

$$\tilde{L}_p(y, \sigma) = L_p(y, \sigma)$$

Here, $\tilde{L}_p$ is well-def'n up to $\mathcal{O}(\mathcal{U})^\times$ & $=$ holds up to $L^\times$.