§3 Ordinary Locus

Def $\mathcal{E}^I_{ord} = \{x \mid v_p(U_p(x)) = 0\}$

Prop $\mathcal{E}^I_{ord}$ is a union of conn'd comp. of $\mathcal{E}^I$, so equidim of dim 1.
The VC points are very Zar dense in $\mathcal{E}^I_{ord}$ & $\kappa : \mathcal{E}^I_{ord} \to \mathcal{W}^I$ is finite.

Proof $U_p \in \mathcal{O}(\mathcal{E}^I)$ is bdd by 1
& unit ball in $\mathcal{O}(\mathcal{E}^I)$ is cpct.
$\Rightarrow e = \lim_{n\to\infty} U_p^{n!}$ exists in $\mathcal{O}(\mathcal{E}^I)$ &
takes value 1 on $\{x : |U_p(x)| = 1\}$
& 0 elsewhere
$\Rightarrow e$ is idempotent & $\mathcal{E}^I_{ord} = e(\mathcal{E}^I) = \{e = 1\}$. $\square$

Prop If $x \in \mathcal{E}^I_{ord}$, then

$$1 \to \underbrace{X_2}_{\substack{\text{HT wgt} \\ -d\kappa(x)}} \to \rho_x|_{G_{\mathbb{Q}_p}} \to \underbrace{X_1}_{\text{unram'd}} \to 1$$

§3 Local Geometry

Let $x \in \mathcal{C}^I(L)$ w/ $\omega = \kappa(x) \in \mathcal{W}(L)$

(Do base change so everything is $/L$ now instead of $/\mathbb{Q}_p$).

Lemma/Def $\exists$ basis of nbgh $\mathcal{U}$ of $x \in \mathcal{C}^I$ s.t.

(a) $\exists$ admissible open $W = \operatorname{Sp} R$ of $\mathcal{W}$ and ν adapted to W s.t.

$$\mathcal{U} \text{ is a C.C. of } x \text{ in } \mathcal{C}^I_{W,\nu}$$

(b) $\kappa^{-1}(\omega) \cap \mathcal{U} = \{x\}$

(c) $\kappa : \mathcal{U} \to W$ is étale (except maybe at x)

Proof $\{\mathcal{C}^I_{W,\nu}\}_{W,\nu}$ is a basis of nbgh of x

Choose $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{C}^I_{W,\nu}$ s.t. $(\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset)$

$$x \in \mathcal{U}_1 \,, \quad \underbrace{(\kappa^{-1}(\omega) \setminus \{x\})}_{\text{finite set}} \subset \mathcal{U}_2$$

We can find admissible $W' \subset \mathcal{W}$ s.t.

$$\mathcal{U}_1 \supset \kappa^{-1}(W') \subset \mathcal{U}_1 \cup \mathcal{U}_2$$

Let $\mathcal{U}$ = C.C. of x in $\kappa^{-1}(W')$

Can choose W' small enough for (c). $\square$

We say such $\mathcal{U}$ is a <u>clean nbgh</u>

$\exists$ idempotent $\varepsilon \in \mathbb{T}^{\Sigma}_{W,\nu}$ s.t.

$\mathcal{U} \subset \mathcal{E}^{\Sigma}_{W,\nu}$ is "$\varepsilon = 1$"

$\rightsquigarrow \mathcal{U} = \mathrm{Sp}(\varepsilon \mathbb{T}^{\pm}_{W,\nu}) = \mathrm{Sp}\, \mathbb{T}$ $(e := \mathrm{rk}\, \mathbb{T})$

Clearly, $\mathbb{T}$ is the eigenalgebra of

$$\underset{\substack{\uparrow \\ \text{finite free, rk } d}}{M} := \varepsilon\, \mathrm{Symb}^{\pm}_{p}(\mathcal{D}_K[r])^{\leq \nu}$$

Let $\kappa : \mathcal{U} \to W$ (restriction), finite map of deg e.

For $\omega' \in W(L)$, $M_{\omega'} = M \otimes_{R,\omega'} L$, $\mathbb{T}_{\omega'} = \mathbb{T} \otimes_{R,\omega'} L$

Then, $\mathbb{T}_{\omega'}$ is eigenalg. of $M_{\omega'}$ (dim $= d$)

for all $\omega' \neq \omega$. At ω, $\mathbb{T}_\omega$ might be larger

<u>Rmk</u> Since $\mathbb{T}_\omega$ is local (its only max ideal is x), we know

$$M_\omega = \underset{\substack{\uparrow \\ \text{gen'l eigensp.}}}{\mathrm{Symb}^{\pm}_{p}(\mathcal{D}_\omega)_{(x)}}$$

Lemma Supp. $\kappa(x) = (z \mapsto z^k \varepsilon(z))$
w/ min. tame lvl $N_0 \mid N$. If $N_0 < N$,
assume x is normal. Then,

$$d = \sigma(N/N_0)\, e$$

Lemma Let $\omega \in W(L)$ be $\omega(z) = z^k \varepsilon(z)$.
Let λ be an $\mathcal{H}$-eigensyst. in $S_p^I(\mathcal{V}_\omega(L))$
which is not E_2.
Let N_0 be its min'l lvl. If $\varepsilon = 1$, assume

$$\lambda(U_p)^2 \neq p^{k+1} \lambda(\langle p \rangle) \qquad (*)$$

$\Rightarrow S_p^I(\mathcal{V}_\omega(L))[\lambda] = S_p^I(\mathcal{V}_\omega(L))_{(\lambda)}$, both
of dim $\sigma(N/N_0)$

THM Let x be a normal classical pt of wgt k s.t. $(*)$ holds.

If x has non-critical slope, then
κ is étale (& smooth) at x

Cor Same holds for $\mathcal{C} \rightsquigarrow \mathcal{C}^I \to \mathcal{C}$
is an $\cong$ in small nbgh of x.

Lemma Let x be normal VC pt of weight k & slope $k+1$ (critical). Then, $\exists ! \rho_x^p : G_{\mathbb{Q},Np} \to GL_2(\mathbb{Q}_p)$ s.t.

(i) $\operatorname{Tr}(\rho_x^p(\mathrm{Frob}_\ell)) = T_\ell(x)$ $(\forall \ell \nmid Np)$

(ii) $(\rho_x^p)|_{G_{\mathbb{Q}_p}}$ is crystalline

(iii) ρ_x^p is indecomposable

Moreover, the H/T weights of $(\rho_x^p)|_{G_{\mathbb{Q}_p}}$ are 0 and $-k-1$ and $U_p(x)$ is an eigenvalue of the crys. Frob on $D_{crys}(\rho_x^p|_{G_{\mathbb{Q}_p}})$.

We say ρ_x^p is the "preferred" Galois rep attached to x

Def We say $x \in \mathcal{E}^{\pm}$ w/ $\kappa(x) = k \in \mathbb{N}$ has a companion pt y if $\exists y \in \mathcal{E}^{\pm(-1)^{k+1}}$ s.t.

(1) $\kappa(y) = -2-k$

(2) $T_\ell(y) = \ell^{-k-1} T_\ell(x)$, $U_p(y) = p^{-k-1} U_p(x)$, $\langle a \rangle(y) = \langle a \rangle(x)$

If y $\exists$'s, it is unique.

THM Let x be a VC point of wgt k. Assume x is not abnormal & of crit. slope. Then, TFAE:

(i) κ is étale at x

(ii) $\rho_x : \mathrm{Symb}_p^{\pm}(\mathcal{D}_k)_{(x)} \xrightarrow{\sim} \mathrm{Symb}_p^{\pm}(\mathcal{D}_k)_{(x)}$

(iii) x has no companion pt.

(iv) $(\rho_x^p)|_{G_{\mathbb{Q}_p}}$ is not $\cong \chi_1 \oplus \chi_2$

THM Let x as above. If x is cusp., assume $H^1_g(\mathbb{Q}, \mathrm{ad}\,\rho_x) = 0$. Then, $\mathcal{E}^{\pm}$ is smooth at x & $\mathcal{E}^{\pm} \to \mathcal{C}$ is an isom. in a small nbgh of x.

Rmk These 2 thm's can be proven together but the proof is very long. The same proof still mostly works for gen'l normal classical points. Need to modify notion of "preferred" rep.

<u>Prop</u> Let g be a mod. form of wgt $k+2$, level $\Gamma = \Gamma_0(p) \cap \Gamma_1(N)$ & crit. slope.
Assume g is an $\mathcal{H}_0$-eigenform.
$\mathcal{H}_0$ = Hecke alg w/o U_p

Then, the unique point $x_g \in \mathcal{C}$ corresp. to g is in $\mathcal{C}^+$ & $\mathcal{C}^-$
If g is cusp $\rightsquigarrow$ x_g is VC in both $\mathcal{C}^\pm$
If g is Eis $\rightsquigarrow$ x_g is VC in $\mathcal{C}^{\varepsilon(g)}$
& not VC in $\mathcal{C}^{-\varepsilon(g)}$

<u>Proof</u> $\exists\, S_{k+2}(\Gamma) \hookrightarrow \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k)$ (for both $\pm$) $\Rightarrow$ Proves it for g cuspidal.
If g is Eisenstein, then Ex. 2.6.15 shows that g critical $\Rightarrow$ $g \in S^\dagger(\Gamma)$.
Hence, $x_g \in \mathcal{C}^0 \subset \mathcal{C}^\pm$. Moreover, it is known that the Eisenstein series has ι-eigenvalue $= \varepsilon(g)$. $\square$

<u>Corollary</u> If g as above then
$$\mathrm{Symb}_\Gamma^+(\mathcal{D}_k)_{(x)} \simeq \mathrm{Symb}_\Gamma^-(\mathcal{D}_k)_{(x)}$$
as $\mathcal{H}_0$-module & $\dim^n = \deg_x(\mathcal{C}^\pm \to \mathcal{W})$

<u>Proof</u> x_g is VC in at least one of $\mathcal{C}^\pm$
$\rightsquigarrow$ Apply prev. thm. $\left(\underset{\text{near } x_g}{\mathcal{C}^+ \simeq \mathcal{C}^-}\right)$ $\square$

Prop Assume x is VC, $\kappa(x)=0$ &
$v_p(U_p(x)) = 1$ (critical).
Suppose $\lambda_{x,0}: \mathcal{H}_0 \to L$ is Eisenstein
$\Rightarrow \kappa$ is étale at x & all the conclusions above hold for x. ($\exists!\, \rho_x^p$, κ étale, $\mathcal{C}^\pm \xrightarrow[x]{\simeq} \mathcal{C}$)

THM Let x be normal classical.
If x is cusp., assume $H^1_g(\mathbb{Q}, \operatorname{ad} \rho_x) = 0$
Then, $\mathcal{C}^\pm$ is smooth at x.
(Proof similar to VC & critical slope).

THM (Bellaïche - Dimitrov)
Let f be a classical cusp form of wgt 1 that is reg. at p ($f_\alpha \neq f_\beta$).
Let $x \in \mathcal{C}^\pm$ be a pt corresp. to one of its refinement $\Rightarrow \mathcal{C}^\pm$ is smooth at x.
Proof Hard, requires Baker's thm on lin. indep. of log (alg. numbers) $\square$

OTOH, if f = unique refinement of a classical cuspidal Eis series of wgt 1 which is irreg at $p \Rightarrow f \in \mathcal{C}^0$ is smooth but belongs to 3 smooth comp. of $\mathcal{C}$ that cross normally.

§Global Properties

Consider any of $\mathcal{C}^\circ, \mathcal{C}^{\pm}$ or $\mathcal{C}$.

For every $h \in \mathcal{H}$, $\exists$ globally defined

$$\det(1 - T\psi(hU_p)) \in \mathcal{O}(\mathcal{W})\{\{T\}\}$$

Prop The coeff. of the Fredholm det

$$\det(1 - T\psi(hU_p))$$

belong to the Iwasawa alg. $\Lambda \subset \mathcal{O}(\mathcal{W})$

Proof h = lin. comb. of T_ℓ, U_p & $\langle a \rangle$

$\Rightarrow \forall \omega \in \mathcal{W}(\mathbb{C}_p)$, hU_p & M_ω has norm ≤ 1.

$\Rightarrow$ coeff $a_n(\omega)$ have abs. values ≤ 1.

By Ex. 6.3.10 $\rightsquigarrow a_n \in \Lambda$. $\square$

This suggest that eigencurves have an "integral model $/\Lambda$". They do, using $\mathrm{spa}(\Lambda, \Lambda)^{an}$.

THM Let $\mathcal{D}$ = open disk $\subset \mathcal{W}$ & $D^* = \mathcal{D} \setminus \{pt\}$

Then, $\exists$ section $s^*: D^* \to \mathcal{C}$ of κ extends to $\mathcal{D}$.

§Open Questions

(1) Are there ∞-many irred. comp of $\mathcal{E}$?

(2) If U is a comp ($U \notin \mathcal{E}^{\circ}_{\mathrm{ord}}$), is it of f. degree over $\mathcal{W}$? (Conj: Yes)

(3) Let $B_r = \bigcup_{i=1}^{p-1} \{ r < |z| < 1 \} \subset \bigcup_{i=1}^{p-1} B(0,1) = \mathcal{W}$

If r is close enough to 1, does $\mathcal{E}$ become étale over B_r?