14. $\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}+\cdots=\sum_{n=1}^{\infty}\frac{1}{3n+2}$. The function $f(x)=\frac{1}{3x+2}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty}\frac{1}{3x+2}\,dx=\lim_{t\to\infty}\int_1^t\frac{1}{3x+2}\,dx=\lim_{t\to\infty}\left[\frac{1}{3}\ln|3x+2|\right]_1^t=\frac{1}{3}\lim_{t\to\infty}(\ln(3t+2)-\ln 5)=\infty$, so the series $\sum_{n=1}^{\infty}\frac{1}{3n+2}$ diverges.

21. $f(x)=\frac{1}{x\ln x}$ is continuous and positive on $[2,\infty)$, and also decreasing since $f'(x)=-\frac{1+\ln x}{x^2(\ln x)^2}<0$ for $x>2$, so we can use the Integral Test. $\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{t\to\infty}[\ln(\ln x)]_2^t=\lim_{t\to\infty}[\ln(\ln t)-\ln(\ln 2)]=\infty$, so the series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges.

24. $f(x)=\frac{x^2}{e^x}\quad\Rightarrow\quad f'(x)=\frac{e^x(2x)-x^2e^x}{(e^x)^2}=\frac{xe^x(2-x)}{(e^x)^2}=\frac{x(2-x)}{e^x}<0$ for $x>2$, so f is continuous, positive, and decreasing on $[3,\infty)$ and so the Integral Test applies.

$\int_3^{\infty}f(x)\,dx=\lim_{t\to\infty}\int_3^t\frac{x^2}{e^x}\,dx\overset{(\star)}{=}\lim_{t\to\infty}\left[-e^{-x}(x^2+2x+2)\right]_3^t=-\lim_{t\to\infty}\left[e^{-t}(t^2+2t+2)-e^{-3}(17)\right]\overset{(\star\star)}{=}\frac{17}{e^3}$,

so the series $\sum_{n=3}^{\infty}\frac{n^2}{e^n}$ converges.

$(\star)$: $\int x^2e^{-x}\,dx\overset{97}{=}-x^2e^{-x}+2\int xe^{-x}\,dx\overset{97}{=}-x^2e^{-x}+2\left(-xe^{-x}+\int e^{-x}\,dx\right)$

$=-x^2e^{-x}-2xe^{-x}-2e^{-x}+C=-e^{-x}(x^2+2x+2)+C$.

$(\star\star)$: $\lim_{t\to\infty}\frac{t^2+2t+2}{e^t}\overset{H}{=}\lim_{t\to\infty}\frac{2t+2}{e^t}\overset{H}{=}\lim_{t\to\infty}\frac{2}{e^t}=0$.

42. For the series $\sum_{n=1}^{\infty}\left(\frac{c}{n}-\frac{1}{n+1}\right)$,

$$s_n=\sum_{i=1}^{n}\left(\frac{c}{i}-\frac{1}{i+1}\right)=\left(\frac{c}{1}-\frac{1}{2}\right)+\left(\frac{c}{2}-\frac{1}{3}\right)+\left(\frac{c}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{c}{n}-\frac{1}{n+1}\right)$$
$$=\frac{c}{1}+\frac{c-1}{2}+\frac{c-1}{3}+\frac{c-1}{4}+\cdots+\frac{c-1}{n}-\frac{1}{n+1}=c+(c-1)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)-\frac{1}{n+1}$$

Thus, $\sum_{n=1}^{\infty}\left(\frac{c}{n}-\frac{1}{n+1}\right)=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left[c+(c-1)\sum_{i=2}^{n}\frac{1}{i}-\frac{1}{n+1}\right]$. Since a constant multiple of a divergent series is divergent, the last limit exists only if $c-1=0$, so the original series converges only if $c=1$.

10. $\frac{n^2-1}{3n^4+1}<\frac{n^2}{3n^4+1}<\frac{n^2}{3n^4}=\frac{1}{3}\frac{1}{n^2}$. $\sum_{n=1}^{\infty}\frac{n^2-1}{3n^4+1}$ converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series $[p=2>1]$. The terms of the given series are positive for $n>1$, which is good enough.

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty} \frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

30. $\dfrac{n!}{n^n} = \dfrac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n}{n \cdot n \cdot n \cdot \cdots \cdot n \cdot n} \le \dfrac{1}{n} \cdot \dfrac{2}{n} \cdot 1 \cdot 1 \cdot \cdots \cdot 1$ for $n \ge 2$, so since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges $[p = 2 > 1]$, $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\displaystyle\lim_{n\to\infty} \frac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n \;\Rightarrow\; \dfrac{1}{n^p \ln n} \ge \dfrac{1}{n \ln n}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 11.3.21), so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \dfrac{1}{n^p \ln n}$ and $b_n = \dfrac{1}{n^p}$. $\displaystyle\sum_{n=2}^{\infty} b_n$ converges, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

44. First we observe that, by l'Hospital's Rule, $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$ by Theorem 11.2.6. Therefore, $\displaystyle\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.