7. Let $u = x^2$, $dv = \sin \pi x\, dx \quad \Rightarrow \quad du = 2x\, dx$ and $v = -\frac{1}{\pi} \cos \pi x$. Then

$I = \int x^2 \sin \pi x\, dx = -\frac{1}{\pi} x^2 \cos \pi x + \frac{2}{\pi} \int x \cos \pi x\, dx$ $(\star)$. Next let $U = x$, $dV = \cos \pi x\, dx \quad \Rightarrow \quad dU = dx$,

$V = \frac{1}{\pi} \sin \pi x$, so $\int x \cos \pi x\, dx = \frac{1}{\pi} x \sin \pi x - \frac{1}{\pi} \int \sin \pi x\, dx = \frac{1}{\pi} x \sin \pi x + \frac{1}{\pi^2} \cos \pi x + C_1$.

Substituting for $\int x \cos \pi x\, dx$ in $(\star)$, we get

$I = -\frac{1}{\pi} x^2 \cos \pi x + \frac{2}{\pi} \left(\frac{1}{\pi} x \sin \pi x + \frac{1}{\pi^2} \cos \pi x + C_1 \right) = -\frac{1}{\pi} x^2 \cos \pi x + \frac{2}{\pi^2} x \sin \pi x + \frac{2}{\pi^3} \cos \pi x + C$, where $C = \frac{2}{\pi} C_1$.

19. Let $u = t$, $dv = \sin 3t\, dt \quad \Rightarrow \quad du = dt$, $v = -\frac{1}{3} \cos 3t$. Then

$\int_0^\pi t \sin 3t\, dt = \left[-\frac{1}{3} t \cos 3t \right]_0^\pi + \frac{1}{3} \int_0^\pi \cos 3t\, dt = \left(\frac{1}{3} \pi - 0 \right) + \frac{1}{9} \left[\sin 3t \right]_0^\pi = \frac{\pi}{3}$.

28. Let $u = (\ln x)^2$, $dv = x^{-3}\, dx \quad \Rightarrow \quad du = \dfrac{2 \ln x}{x}\, dx$, $v = -\frac{1}{2} x^{-2}$. Then

$I = \displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\, dx = \left[-\frac{(\ln x)^2}{2x^2} \right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\, dx$. Now let $U = \ln x$, $dV = x^{-3}\, dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\, dx$, $V = -\frac{1}{2} x^{-2}$.

Then

$$\int_1^2 \frac{\ln x}{x^3}\, dx = \left[-\frac{\ln x}{2x^2} \right]_1^2 + \frac{1}{2} \int_1^2 x^{-3} dx = -\frac{1}{8} \ln 2 + 0 + \frac{1}{2} \left[-\frac{1}{2x^2} \right]_1^2 = -\frac{1}{8} \ln 2 + \frac{1}{2} \left(-\frac{1}{8} + \frac{1}{2} \right) = \frac{3}{16} - \frac{1}{8} \ln 2.$$

Thus $I = \left(-\frac{1}{8} (\ln 2)^2 + 0 \right) + \left(\frac{3}{16} - \frac{1}{8} \ln 2 \right) = -\frac{1}{8} (\ln 2)^2 - \frac{1}{8} \ln 2 + \frac{3}{16}$.

48. Let $u = x^n$, $dv = e^x\, dx \quad \Rightarrow \quad du = n x^{n-1}\, dx$, $v = e^x$. By Equation 2, $\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$.

52. By repeated applications of the reduction formula in Exercise 48,

$$\begin{aligned} \int x^4 e^x\, dx &= x^4 e^x - 4 \int x^3 e^x\, dx = x^4 e^x - 4 \left(x^3 e^x - 3 \int x^2 e^x\, dx \right) \\ &= x^4 e^x - 4x^3 e^x + 12 \left(x^2 e^x - 2 \int x^1 e^x\, dx \right) = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 \left(x^1 e^x - \int x^0 e^x\, dx \right) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 e^x + C \quad \left[\text{or } e^x (x^4 - 4x^3 + 12x^2 - 24x + 24) + C \right] \end{aligned}$$

2.
$$\begin{aligned} \int \sin^6 x \cos^3 x\, dx &= \int \sin^6 x \cos^2 x \cos x\, dx = \int \sin^6 x \left(1 - \sin^2 x \right) \cos x\, dx \overset{s}{=} \int u^6 (1 - u^2)\, du \\ &= \int (u^6 - u^8)\, du = \tfrac{1}{7} u^7 - \tfrac{1}{9} u^9 + C = \tfrac{1}{7} \sin^7 x - \tfrac{1}{9} \sin^9 x + C \end{aligned}$$

12. Let $u = x$, $dv = \cos^2 x\, dx \quad \Rightarrow \quad du = dx$, $v = \int \cos^2 x\, dx = \int \frac{1}{2} (1 + \cos 2x)\, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x$, so

$$\begin{aligned} \int x \cos^2 x\, dx &= x \left(\tfrac{1}{2} x + \tfrac{1}{4} \sin 2x \right) - \int \left(\tfrac{1}{2} x + \tfrac{1}{4} \sin 2x \right) dx = \tfrac{1}{2} x^2 + \tfrac{1}{4} x \sin 2x - \tfrac{1}{4} x^2 + \tfrac{1}{8} \cos 2x + C \\ &= \tfrac{1}{4} x^2 + \tfrac{1}{4} x \sin 2x + \tfrac{1}{8} \cos 2x + C \end{aligned}$$

30. $\int_0^{\pi/3} \tan^5 x \, \sec^6 x \, dx = \int_0^{\pi/3} \tan^5 x \, \sec^4 x \, \sec^2 x \, dx = \int_0^{\pi/3} \tan^5 x \, (1 + \tan^2 x)^2 \, \sec^2 x \, dx$

$$= \int_0^{\sqrt{3}} u^5 (1+u^2)^2 \, du \quad [u = \tan x, \, du = \sec^2 x \, dx] \quad = \int_0^{\sqrt{3}} u^5 (1 + 2u^2 + u^4) \, du$$

$$= \int_0^{\sqrt{3}} (u^5 + 2u^7 + u^9) \, du = \left[\tfrac{1}{6}u^6 + \tfrac{1}{4}u^8 + \tfrac{1}{10}u^{10}\right]_0^{\sqrt{3}} = \tfrac{27}{6} + \tfrac{81}{4} + \tfrac{243}{10} = \tfrac{981}{20}$$

Alternate solution:

$$\int_0^{\pi/3} \tan^5 x \, \sec^6 x \, dx = \int_0^{\pi/3} \tan^4 x \, \sec^5 x \, \sec x \, \tan x \, dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \, \sec^5 x \, \sec x \, \tan x \, dx$$

$$= \int_1^2 (u^2 - 1)^2 u^5 \, du \qquad [u = \sec x, \, du = \sec x \, \tan x \, dx]$$

$$= \int_1^2 (u^4 - 2u^2 + 1) u^5 \, du = \int_1^2 (u^9 - 2u^7 + u^5) \, du$$

$$= \left[\tfrac{1}{10}u^{10} - \tfrac{1}{4}u^8 + \tfrac{1}{6}u^6\right]_1^2 = \left(\tfrac{512}{5} - 64 + \tfrac{32}{3}\right) - \left(\tfrac{1}{10} - \tfrac{1}{4} + \tfrac{1}{6}\right) = \tfrac{981}{20}$$

36. $\displaystyle\int \frac{\sin\phi}{\cos^3\phi} \, d\phi = \int \frac{\sin\phi}{\cos\phi} \cdot \frac{1}{\cos^2\phi} \, d\phi = \int \tan\phi \, \sec^2\phi \, d\phi = \int u \, du \qquad [u = \tan\phi, \, du = \sec^2\phi \, d\phi]$

$$= \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\tan^2\phi + C$$

Alternate solution: Let $u = \cos\phi$ to get $\frac{1}{2}\sec^2\phi + C$.

70. $\displaystyle\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, \sin mx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\left(\sum_{n=1}^{m} a_n \sin nx\right) \sin mx\right] dx = \sum_{n=1}^{m} \frac{a_n}{\pi} \int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx$. By Exercise 68, every term is zero except the mth one, and that term is $\dfrac{a_m}{\pi} \cdot \pi = a_m$.