4. Since $f^{(n)}(4) = \dfrac{(-1)^n\, n!}{3^n(n+1)}$, Equation 6 gives the Taylor series

$\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{3^n(n+1)\, n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered at 4. Apply the Ratio Test to find the radius of convergence R.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)}\cdot\frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right|$$

$$= \frac{1}{3}\,|x-4| \lim_{n\to\infty}\frac{n+1}{n+2} = \frac{1}{3}\,|x-4|$$

For convergence, $\frac{1}{3}\,|x-4| < 1 \quad\Leftrightarrow\quad |x-4| < 3$, so $R = 3$.

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
⋮	⋮	⋮

$$\ln(1+x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$

$$= 0 + x - \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 - \tfrac{6}{24}x^4 + \tfrac{24}{120}x^5 - \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/n} = |x| < 1$ for convergence, so $R = 1$.

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
⋮	⋮	⋮

$$e^{5x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{5^{n+1}\,|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n\,|x|^n}\right] = \lim_{n\to\infty}\frac{5\,|x|}{n+1} = 0 < 1$$

for all x, so $R = \infty$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right]$$

$$= \lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x\text{, so } R = \infty.$$

28. $$(1-x)^{2/3} = \sum_{n=0}^{\infty}\binom{\frac{2}{3}}{n}(-x)^n = 1 + \tfrac{2}{3}(-x) + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!}(-x)^2 + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(-x)^3 + \cdots$$

$$= 1 - \tfrac{2}{3}x + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot 2\cdot[1\cdot 4\cdot 7\cdot\cdots\cdot(3n-5)]}{3^n\cdot n!}x^n$$

$$= 1 - \tfrac{2}{3}x - 2\sum_{n=2}^{\infty}\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-5)}{3^n\cdot n!}x^n$$

and $|-x| < 1 \quad\Leftrightarrow\quad |x| < 1$, so $R = 1$.

36. $$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1 + \left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$

$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \quad\Leftrightarrow\quad |x| < 2, \quad \text{so } R = 2.$$

48. $$e^x \overset{(11)}{=} \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad\Rightarrow\quad e^x - 1 = \sum_{n=1}^{\infty}\frac{x^n}{n!} \quad\Rightarrow\quad \frac{e^x-1}{x} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} \quad\Rightarrow\quad \int\frac{e^x-1}{x}\,dx = C + \sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!},$$

with $R = \infty$.

56. $\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$

$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots} = \frac{\frac{1}{2}-0}{-\frac{1}{2}-0} = -1$$

since power series are continuous functions.

62. From Example 6 in Section 11.9, we have $\ln(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$$e^x\ln(1-x) = \left(1+x+\tfrac{1}{2}x^2+\cdots\right)\left(-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\cdots\right)$$

$$= -x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-x^2-\tfrac{1}{2}x^3-\tfrac{1}{2}x^3-\cdots = -x-\tfrac{3}{2}x^2-\tfrac{4}{3}x^3-\cdots,\ |x|<1$$

64. $\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, by (16).