18. If $a_n = \frac{n}{4^n}(x+1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(x+1)^n}\right| = \frac{|x+1|}{4}\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x+1|}{4}$.

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{n}{4^n}(x+1)^n$ converges when $\frac{|x+1|}{4} < 1 \quad\Leftrightarrow\quad |x+1| < 4 \quad [R = 4] \quad\Leftrightarrow$

$-4 < x+1 < 4 \quad\Leftrightarrow\quad -5 < x < 3$. When $x = -5$ or 3, both series $\sum_{n=1}^{\infty}(\mp 1)^n\, n$ diverge by the Test for Divergence since

$\lim_{n\to\infty}|(\mp 1)^n\, n| = \infty$. Thus, the interval of convergence is $I = (-5, 3)$.

27. If $a_n = \frac{x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)} \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{2n+1} = 0 < 1$. Thus, by the

Ratio Test, the series $\sum_{n=1}^{\infty}\frac{x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$ converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

28. If $a_n = \frac{n!\, x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$, then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\, x^{n+1}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)} \cdot \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\, x^n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x|}{2n+1} = \frac{1}{2}|x|$.

By the Ratio Test, the series $\sum_{n=1}^{\infty} a_n$ converges when $\frac{1}{2}|x| < 1 \quad\Rightarrow\quad |x| < 2$, so $R = 2$. When $x = \pm 2$,

$|a_n| = \frac{n!\, 2^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \frac{[1\cdot 2\cdot 3\cdot\cdots\cdot n]\, 2^n}{[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]} = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} > 1$, so both endpoint series

diverge by the Test for Divergence. Thus, the interval of convergence is $I = (-2, 2)$.

6. $f(x) = \frac{1}{x+10} = \frac{1}{10}\left(\frac{1}{1-(-x/10)}\right) = \frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{10}\right)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n\frac{1}{10^{n+1}}x^n$. The series converges

when $\left|\frac{x}{10}\right| < 1$, that is, when $|x| < 10$, so $R = 10$ and $I = (-10, 10)$.

8. $f(x) = \frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$. The series converges when

$|-2x^2| < 1 \quad\Rightarrow\quad |x^2| < \frac{1}{2} \quad\Rightarrow\quad |x| < \frac{1}{\sqrt{2}}$, so $R = \frac{1}{\sqrt{2}}$ and $I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

14. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$ [geometric series with $R = 1$], so

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

[$C = 0$ since $f(0) = \ln 1 = 0$], with $R = 1$

(b) $f(x) = x \ln(1+x) = x\left[\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^{n+1}}{n} = \sum_{n=2}^{\infty} \dfrac{(-1)^n x^n}{n-1}$ with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} (x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^{2n}}{n}$ with $R = 1$.

15. $f(x) = \ln(5-x) = -\displaystyle\int \frac{dx}{5-x} = -\frac{1}{5}\int \frac{dx}{1-x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right] dx = C - \frac{1}{5}\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty} \frac{x^n}{n\,5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \Leftrightarrow |x| < 5$, so $R = 5$.

18. From Example 7, $g(x) = \arctan x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3, \text{ so } R = 3.$$

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty} \dfrac{t^n}{n}$ for $|t| < 1$, so $\dfrac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \dfrac{t^{n-1}}{n}$ and $\displaystyle\int \frac{\ln(1-t)}{t}\, dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$.

By Theorem 2, $R = 1$.

32. $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \dfrac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad \text{[substituting } n+1 \text{ for } n\text{]}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \Rightarrow f''(x) + f(x) = 0.$$