8. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!}\right] = \lim_{n\to\infty}\frac{n+1}{100} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges by the Ratio Test.

12. $\left|\frac{\sin 4n}{4^n}\right| \le \frac{1}{4^n}$, so $\sum_{n=1}^{\infty}\left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty}\frac{1}{4^n}$ $\left[|r| = \frac{1}{4} < 1\right]$.

Thus, $\sum_{n=1}^{\infty}\frac{\sin 4n}{4^n}$ is absolutely convergent.

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^2\, 2^{n+1}}{(n+1)!}\cdot\frac{n!}{n^2 2^n}\right] = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^2\cdot\frac{2}{n+1}\right] = 0$, so the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

20. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|} = \lim_{n\to\infty}\frac{2}{n} = 0 < 1$, so the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

24. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\frac{n^{1/n}}{\ln n} \overset{(\star)}{=} 0 < 1$, so the series $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^n}$ is absolutely convergent by the Root Test.

$(\star)$ Let $y = x^{1/x}$. Then $\ln y = \frac{1}{x}\ln x$, so $\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0 \;\Rightarrow\; \lim_{x\to\infty} y = e^0 = 1$.

10. If $a_n = \frac{10^n\, x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}\, x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n\, x^n}\right| = \lim_{n\to\infty}\left|\frac{10x\, n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\frac{10\,|x|}{(1+1/n)^3} = \frac{10\,|x|}{1^3} = 10\,|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{10^n\, x^n}{n^3}$ converges when $10\,|x| < 1 \;\Leftrightarrow\; |x| < \frac{1}{10}$, so the radius of convergence is $R = \frac{1}{10}$. When $x = -\frac{1}{10}$, the series converges by the Alternating Series Test; when $x = \frac{1}{10}$, the series converges because it is a p-series with $p = 3 > 1$. Thus, the interval of convergence is $I = \left[-\frac{1}{10}, \frac{1}{10}\right]$.

14. $a_n = (-1)^n\frac{x^{2n}}{(2n)!}$, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty}\frac{|x|^2}{(2n+1)(2n+2)} = 0 < 1$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

23. If $a_n = n!\,(2x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$ for all $x \ne \frac{1}{2}$. Since the series diverges for all $x \ne \frac{1}{2}$, $R = 0$ and $I = \left\{\frac{1}{2}\right\}$.

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n (-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.

33. No. If a power series is centered at a, its interval of convergence is symmetric about a. If a power series has an infinite radius of convergence, then its interval of convergence must be $(-\infty, \infty)$, not $[0, \infty)$.