Practice Final Exam

- PART I. Techniques of Integration and Improper Integrals.

  1. Compute the following integrals.

     (a) $\int \ln(x^2 - 1)\, dx$

     Solution. Use integration by parts with $f = \ln(x^2 - 1), g' = 1$, then

     $$\int \ln(x^2 - 1)\, dx = x \ln(x^2 - 1) - \int \frac{2x^2}{x^2 - 1} dx.$$

     Now compute, (use long division)

     $$\int 2\frac{x^2}{x^2 - 1} dx = \int (2 + 2\frac{1}{x^2 - 1}) dx.$$

     By partial fraction decomposition,

     $$\frac{2}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1}.$$

     So,

     $$\int 2\frac{1}{x^2 - 1} dx = \int \frac{1}{x - 1} dx - \int \frac{1}{x + 1} dx = \ln|x-1| - \ln|x+1|.$$

     Putting all the pieces together,

     $$\int \ln(x^2 - 1)\, dx = x \ln(x^2 - 1) - 2x - \ln|x - 1| + \ln|x + 1| + C.$$

(b) $\int \sqrt{6 + 4x - x^2}\, dx$

Solution. Complete the square:

$$6 + 4x - x^2 = 10 - (x-2)^2$$

then set Set $x - 2 = \sqrt{10}\sin\theta$ then $dx = \sqrt{10}\cos\theta d\theta$ and $\sqrt{10 - (x-2)^2} = \sqrt{10}\cos\theta$

$$\int \sqrt{6 + 4x - x^2}\, dx = 10 \int \cos^2\theta d\theta =$$

$$= 10 \int \frac{1 + \cos 2\theta}{2} d\theta = 10(\theta/2 + \sin 2\theta/4) + C$$

Since $x - 2 = \sqrt{10}\sin\theta$ we get

$$\sin\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - \sin^2\theta} = \frac{1}{5}(x-2)\sqrt{10 - (x-2)^2}$$

and we can substitute it in to obtain the final answer.

2. Determine whether the following integrals are converging or diverging.

   (a) $\displaystyle\int_1^{+\infty} \frac{e^x}{\sqrt{x}+2}\,dx$

   Solution. Notice that if $x \geq 1$, then $e^x \geq 1$ and $\sqrt{x}+2 \leq 3\sqrt{x}$. Then,

   $$\frac{e^x}{\sqrt{x}+2} \geq \frac{1}{3\sqrt{x}}$$

   and our integral diverges by comparison with a p-integral with $p = 1/2$.

   (b) $\displaystyle\int_0^1 \frac{\ln x}{x^3}\,dx$

   Solution.

   $$\int_0^1 \frac{\ln x}{x^3}\,dx = \lim_{t\to 0}\int_t^1 \frac{\ln x}{x^3}\,dx.$$

   Compute (integration by parts with $f = \ln x, g = 1/x^3$)

   $$\int \frac{\ln x}{x^3}\,dx = -\frac{1}{2x^2}\ln x + \int \frac{1}{2x^3}dx = -\frac{1}{2x^2}\ln x - \frac{1}{4x^2}.$$

   So,

   $$\int_0^1 \frac{\ln x}{x^3}\,dx = \lim_{t\to 0}\int_t^1 \frac{\ln x}{x^3}\,dx = \lim_{t\to 0}(-1/4+\frac{1}{4t^2}(2\ln t+1)) = -\infty$$

   and the given integral is divergent.

3. Mark each of the following statements as "T" if they are always true, otherwise mark as "F".

   (a) T F

   $$\int_1^\infty \frac{1}{x^{\sqrt{2}}} dx$$

   is convergent. T

   (b) T F Using partial fractions, the function

   $$\frac{x^3+4}{x(x^2+4)^2}$$

   can be put in the form

   $$\frac{A}{x} + \frac{Bx+C}{x^2+4}$$

   F

   (c) T F Using partial fractions, the function

   $$\frac{x^2-1}{x(x^2+x)}$$

   can be put in the form

   $$\frac{A}{x} + \frac{Bx+C}{x^2+x}$$

   F

- Part II. Numerical series.

1. (a) Find the sum of the following series.

$$\sum_{n=1}^{\infty}(e^{1/(n+2)} - e^{1/n}).$$

Solution. Compute the partial sums:

$$S_1 = e^{1/3} - e, S_2 = e^{1/4} - e^{1/2} + e^{1/3} - e$$
$$S_3 = e^{1/5} - e^{1/3} + e^{1/4} - e^{1/2} + e^{1/3} - e = e^{1/5} + e^{1/4} - e^{1/2} - e$$
$$S_4 = e^{1/5} - e^{1/2} - e + e^{1/6}$$

In general

$$S_n = e^{1/(n+2)} + e^{1/(n+1)} - e^{1/2} - e \to 2 - e^{1/2} - e.$$

(b) Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}.$$

Solution. Apply the integral test. Set $f(x) = \frac{1}{x\sqrt{\ln x}}$. Compute that $f'(x) \leq 0$. Then compute

$$\int_2^{\infty}\frac{1}{x\sqrt{\ln x}} = \lim_{t\to+\infty}\int_2^t \frac{1}{x\sqrt{\ln x}} =$$
$$= \lim_{t\to+\infty} 2\sqrt{\ln x}|_2^t = +\infty.$$

Thus, the series diverges by the integral test.

2. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+7}{(1+4n^2)^4}$

   Apply the limit comparison test with $b_n = 1/n^7$ to conclude that this series converges.

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$

   Apply the alternating series test to conclude that this series converges. Show that $b_n = \frac{e^{1/n}}{n} \to 0$ and b_n is decreasing (set $f(x) = e^{1/x}/x$ and show that $f' < 0$.)

3. Determine whether the series is conditionally convergent, absolutely convergent or divergent.

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$

Apply the ratio test. Set

$$a_n = (-1)^{n+1}\frac{n^2 2^n}{n!}.$$

Then,

$$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^2 2^{n+1}}{(n+1)!}\frac{n!}{n^2 2^n} = \\ = 2(n+1)/n^2 \to 0 < 1.$$

The series is absolutely convergent by the ratio test.

(b) $\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$

Set

$$a_n = \frac{\cos(n\pi/3)}{n!}.$$

Then

$$|a_n| \leq \frac{1}{n!}.$$

Since $\sum_n \frac{1}{n!}$ converges (apply the ratio test), then also $\sum_n |a_n|$ converges by the comparison test. Thus, $\sum_n a_n$ is absolutely convergent.

- Part III. Power series and Taylor series.

  1. Find the radius of convergence and interval of convergence of the series
  $$\sum_{n=1}^{\infty} \frac{n(x-4)^n}{2n^3+3}.$$
  Set
  $$a_n = \frac{n(x-4)^n}{2n^3+3}.$$
  Compute,
  $$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)}{2(n+1)^3+3} \frac{2n^3+3}{n} |x-4| \to |x-4|.$$
  Thus $R = 1$ and the series converges for $|x-4| < 1$ i.e $3 < x < 5$. When $|x-4| = 1$, then
  $$\sum_n |a_n| = \sum_n \frac{n}{2n^3+3}$$
  which converges using the limit comparison test with $b_n = 1/n^2$. Thus, the interval of convergence is $[3, 5]$.

2. Use differentiation and/or integration term by term, to determine a power series representation for the function $f(x) = \arctan(x^2/4)$. Determine also the associated radius of convergence.

   Solution.

$$f'(x) = \frac{x}{2}\frac{1}{1+x^4/16} = \frac{x}{2}\sum_{n=0}^{\infty}(-\frac{x^4}{16})^n, \quad |x|^4 < 16.$$

So,

$$f(x) = \int f'(x)dx = \frac{1}{2}\int \sum_n (-1)^n \frac{x^{4n+1}}{16^n} dx =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(4n+2)(16)^n} + C,$$

and since $f(0) = 0$ we find that $C = 0$ and

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(4n+2)(16)^n},$$

when $|x| < 2$ (and the radius of convergence is then $R = 2$.)
(You can also compute the series expansion for $\arctan y$ and then replace y with $x^2/4$.)

3. Use the definition to find the Taylor series of $f(x) = \sin 3x$ centered at 0. Also find the associated radius of convergence.

   Solution. Compute,

$$f(x) = \sin 3x, f'(x) = 3\cos 3x, f''(x) = -3^2 \sin 3x,$$
$$f'''(x) = -3^3 \cos 3x, f^{(4)}(x) = 3^4 \sin 3x.$$

Thus,

$$f(0) = f''(0) = f^{(4)}(0) = ... = f^{(2k)}(0) = 0$$

while

$$f'(0) = 3, f'''(0) = -3^3 =, ...f^{(2k+1)}(0) = (-1)^k 3^{2k+1}.$$

So the Taylor series for f centered at 0 is given by,

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!} x^{2n+1}.$$

To find R, set

$$a_n = (-1)^n \frac{3^{2n+1}}{(2n+1)!} x^{2n+1}$$

and compute

$$|\frac{a_{n+1}}{a_n}| = \frac{3^2 x^2}{(2n+2)(2n+3)} \to 0$$

thus $R = \infty$.