

## Practice Midterm 2

1. Determine whether the following integrals are converging or diverging. Justify your answer.

$$(a) \int_1^{+\infty} \frac{2}{x(e^{-x} + 1)} dx$$

Since  $e^{-x} \leq 1$  for  $x \geq 0$ , we have that

$$\frac{2}{x(e^{-x} + 1)} \geq \frac{2}{x(1 + 1)} = \frac{1}{x}.$$

We know that

$$\int_1^{+\infty} \frac{1}{x}$$

diverges, thus by the comparison test also the given integral diverges.

$$(b) \int_5^{+\infty} \frac{3}{x \ln x} dx$$

$$\int_5^{+\infty} \frac{3}{x \ln x} dx = 3 \lim_{t \rightarrow +\infty} \int_5^t \frac{3}{x \ln x} dx = (\text{substitute } u = \ln x)$$

$$3 \lim_{t \rightarrow +\infty} \int_{\ln 5}^{\ln t} \frac{du}{u} = 3 \lim_{t \rightarrow \infty} (\ln(\ln(t)) - \ln(\ln(5))) = +\infty.$$

Hence the integral is diverging.

2. Determine whether the sequence converges or diverges. If it converges, find the limit. Justify your answer.

(a)  $a_n = \frac{\cos(3n)}{1 + n^2}$

We know that  $|\cos 3n| \leq 1$ , thus

$$0 \leq |a_n| \leq \frac{1}{1 + n^2} \rightarrow 0 \text{ as } n \rightarrow +\infty$$

By the squeeze theorem  $|a_n| \rightarrow 0$ . Hence  $a_n$  converges to 0.

(b)  $a_n = \ln(2n^2 + 1) - \ln(n^2 + 7)$

By the properties of the logarithmic function we have

$$a_n = \ln\left(\frac{2n^2 + 1}{n^2 + 7}\right).$$

Since

$$\frac{2n^2 + 1}{n^2 + 7} \rightarrow 2$$

(ratio of polynomials of the same degree), then

$$a_n \rightarrow \ln 2.$$

3. Determine whether the series is convergent or divergent. Justify your answer.

(a) 
$$\sum_{n=1}^{\infty} \frac{e^n}{n^{100}}$$

Since

$$e^n/n^{100} \rightarrow \infty$$

(by a repeated application of L'Hopital's rule), we obtain that our series diverges by the Divergence test.

(b) 
$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 10}}{2n^3 + n^2 + 5}$$

Set

$$b_n = \frac{n}{2n^3} = \frac{1}{2n^2}.$$

Then

$$\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = \lim_{n \rightarrow +\infty} \frac{2\sqrt{n^6 + 10n^4}}{2n^3 + n^2 + 5} = 1 > 0$$

(the highest power in the numerator and denominator are the same, thus the limit is the ratio of the coefficients i.e. 2/2.)

We know that  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges ( $p$ -series with  $p = 2$ ), so by the limit comparison test, our series is convergent as well.

4. Find the sum of the series.

$$\begin{aligned} \text{(a)} \quad \sum_{n=1}^{\infty} \frac{(-5)^{n-1}}{3^{3n-4}} \\ \sum_{n=1}^{\infty} \frac{(-5)^{n-1}}{3^{3n-4}} &= 3^4 \sum_{n=1}^{\infty} \frac{(-5)^{n-1}}{(3^3)^n} = 3 \sum_{n=1}^{\infty} \frac{(-5)^{n-1}}{(3^3)^{n-1}} \\ &= 3 \sum_{n=1}^{\infty} \left(\frac{-5}{3^3}\right)^{n-1} = \frac{3}{1 + \frac{5}{27}} = \frac{81}{32} \end{aligned}$$

$$\text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$\begin{aligned} S_k &= \sum_{n=1}^k \frac{1}{(n+1)(n+2)} = \sum_{n=1}^k \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{k+1} - \frac{1}{k+2} = \\ &= \frac{1}{2} - \frac{1}{k+2}. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \lim_{k \rightarrow \infty} S_k = 1/2.$$

5. Determine whether the series is convergent or divergent. Justify your answer.

(a) 
$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

Set  $f(x) = \frac{x}{e^{x^2}}$ . Then  $f$  is a positive decreasing function on  $[1, +\infty)$  ( $f'(x) = e^{-x^2}(1 - 2x^2) < 0$ .) Hence we can apply the integral test. Compute, (make the substitution  $u = -x^2$ )

$$\int \frac{x}{e^{x^2}} dx = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^{-x^2}.$$

Hence

$$\int_1^{\infty} \frac{x}{e^{x^2}} dx = -\frac{1}{2} \lim_{t \rightarrow \infty} (e^{-t^2} - e^{-1}) = \frac{e^{-1}}{2}.$$

Thus, our series converges.

(b) 
$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^4 + 1}}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^4 + 1}} = \sum_{n=1}^{\infty} (-1)^n b_n,$$

$b_n = \frac{n}{\sqrt{n^4 + 1}}$  is decreasing. Indeed, set  $f(x) = \frac{x}{\sqrt{x^4 + 1}}$ . Then,

$$f'(x) = \frac{\sqrt{x^4 + 1} - x \frac{2x^3}{\sqrt{x^4 + 1}}}{x^4 + 1} = \frac{1 - x^4}{(x^4 + 1)^{3/2}} < 0, \quad x > 1.$$

Also,  $b_n \rightarrow 0$ , as  $n \rightarrow +\infty$  (the degree of the denominator is higher than the degree of the numerator.)

By Alternating Series Test, the series converges.