Midterm 2 – Solutions (Sketches)

1. Determine whether the following integrals are converging or diverging. Justify your answer.

   (a) $\displaystyle\int_1^{+\infty} \frac{\sin^2 x + 3}{x^3 + 2x^2 + 1}\, dx$

   Solution. Since
   $$\frac{\sin^2 x + 3}{x^3 + 2x^2 + 1} \leq \frac{4}{x^3}$$
   this integral converges by comparison with a p-integral with $p = 3 > 1$ (hence convergent.)

   (b) $\displaystyle\int_1^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$

   Solution. This integral is convergent since
   $$\int_1^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t\to+\infty} \int_1^{t} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx =$$
   $$= \lim_{t\to+\infty} -2e^{-\sqrt{x}}|_1^t = 2/e.$$

2. Determine whether the sequence converges or diverges. If it converges, find the limit. Justify your answer.

   I do not include solutions to these problems since they are not on the final.

   (a) $a_n = \dfrac{(-4)^n}{n!}$

   (b) $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

3. Find the sum of the series.

   (a) $\displaystyle\sum_{n=1}^{\infty}\left[\cos\frac{1}{n^2}-\cos\frac{1}{(n+1)^2}\right]$

   Solution. Compute the partial sums,

   $$S_k=\sum_{n=1}^{k}\left[\cos\frac{1}{n^2}-\cos\frac{1}{(n+1)^2}\right]=\cos 1-\cos\frac{1}{(k+1)^2}.$$

   Taking the limit we get that the sum of the series is $\cos 1-1$.

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^n}{4^{n+1}}$

   Solution.

   $$\sum_{n=0}^{\infty}\frac{\pi^n}{4^{n+1}}=\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{\pi}{4}\right)^n=\frac{1}{4}\frac{1}{1-(\pi/4)}.$$

4. Determine whether the series is convergent or divergent. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

Solution. Since

$$\ln\left(\frac{n^2+1}{2n^2+1}\right) \to \ln(1/2) \neq 0$$

this series diverges by the divergence test.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

Solution. Apply the integral test. Set $f(x) = \frac{\ln x}{x^2}$. Show that $f' \leq 0$. Then compute

$$\int_2^{\infty} \frac{\ln x}{x^2} dx = \lim_{t\to+\infty} \int_2^t \frac{\ln x}{x^2} dx = (IBP)$$

$$\lim_{t\to+\infty} -\frac{1}{x}\ln x - \frac{1}{x}|_2^t = \ln 2/2 + 1/2.$$

Thus this series is convergent by the integral test.

5. Determine whether the series is convergent or divergent. Justify your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{\frac{1}{n}}}{n}$

   Solution. Apply the alternating series test to conclude that the series converges. Show that $b_n = \frac{e^{\frac{1}{n}}}{n}$ satisfies

   $$b_n \to 0 \quad b_n \geq b_{n+1}.$$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{\sqrt[3]{n^7+n^2}}$

   Solution. Apply the limit comparison test with $b_n = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}}$ (p-series with $p = 4/3 > 1$ hence convergent) to conclude that this series is convergent.