Midterm 1–Solutions (Sketches)

1. Calculate the following integral.

$$\int e^{-x}\cos(2x)\,dx$$

Solutions. Apply IBP twice:

$$\int e^{-x}\cos(2x)\,dx = e^{-x}\frac{1}{2}\sin(2x) + \int e^{-x}\frac{1}{2}\sin(2x)dx =$$
$$= e^{-x}\frac{1}{2}\sin(2x) - e^{-x}\frac{1}{4}\cos(2x) - \frac{1}{4}\int e^{-x}\cos(2x)dx.$$

Thus,

$$\frac{5}{4}\int e^{-x}\cos(2x)dx = e^{-x}\frac{1}{2}\sin(2x) - e^{-x}\frac{1}{4}\cos(2x)$$

and hence

$$\int e^{-x}\cos(2x)dx = \frac{4}{5}(e^{-x}\frac{1}{2}\sin(2x) - e^{-x}\frac{1}{4}\cos(2x)) + C$$

2. Calculate the following integral.

$$\int \tan^{-1}\left(\frac{1}{x}\right)\, dx$$

Solution. Apply IBP with $g' = 1$:

$$\begin{aligned}\int \tan^{-1}\left(\frac{1}{x}\right)\, dx &= x\tan^{-1}\left(\frac{1}{x}\right) - \int x\frac{1}{1+\frac{1}{x^2}}(-\frac{1}{x^2})dx = \\ &= x\tan^{-1}\left(\frac{1}{x}\right) + \int \frac{x}{x^2+1}dx = \\ &= x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln(x^2+1) + C.\end{aligned}$$

3. Calculate the following integral.

$$\int \frac{\cos x}{(\sin x - 2)(\sin^2 x + 4)} dx$$

Solution. Set $u = \sin x$ then the integral becomes

$$\int \frac{du}{(u-2)(u^2+4)}.$$

By partial fractions:

$$\frac{1}{(u-2)(u^2+4)} = \frac{A}{u-2} + \frac{Bu+C}{u^2+4}$$

which gives

$$A = 1/8, B = -1/8, C = -1/2.$$

Thus,

$$\int \frac{du}{(u-2)(u^2+4)} = \frac{1}{8}\ln|u-2| - \frac{1}{8}\int \frac{u+4}{u^2+4} du =$$

$$= \frac{1}{8}\ln|u-2| - \frac{1}{16}\ln(u^2+4) - \frac{1}{8}\tan^{-1}(u/2) + C.$$

4. Calculate the following integral.

$$\int \sqrt{5+4x-x^2}dx$$

Solution.

$$\int \sqrt{5+4x-x^2}dx = \int \sqrt{9-(x-2)^2}dx.$$

Set $x-2 = 3\sin\theta$, then $dx = 3\cos\theta d\theta$ and $\sqrt{9-(x-2)^2} = 3\cos\theta$. Thus,

$$\int \sqrt{5+4x-x^2}dx = 9\int \cos^2\theta d\theta = \frac{9}{2}\int \frac{1+\cos 2\theta}{2}d\theta =$$

$$= \frac{9}{4}\theta + \frac{9}{8}\sin 2\theta + C.$$

Replace $\theta = \sin^{-1}(\frac{x-2}{3})$ and $\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{x-2}{3}\frac{\sqrt{9-(x-2)^2}}{3}$.

5. Calculate the following integral.

$$\int x^3 e^{-x^2}\, dx$$

Solution. Set $u = x^2$, then integrate by parts:

$$\int x^3 e^{-x^2}\, dx = \frac{1}{2}\int u e^{-u} du = \frac{1}{2}(-ue^{-u} - e^{-u}) + C.$$