1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2-5x+6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1,2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

6. $\displaystyle\int_{-\infty}^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln 5 - \tfrac{1}{2}\ln|2t-5|\right] = -\infty.$

Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx.$

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim\limits_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim\limits_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left(1-e^{-t^2}\right) = -\tfrac{1}{2}\cdot 1 = -\tfrac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim\limits_{t\to\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim\limits_{t\to\infty}\left(-\tfrac{1}{2}\right)\left(e^{-t^2}-1\right) = -\tfrac{1}{2}\cdot(-1) = \tfrac{1}{2}.$

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\tfrac{1}{2}+\tfrac{1}{2} = 0.$ Convergent

19. $\displaystyle\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\tfrac{1}{5}se^{-5s} - \tfrac{1}{25}e^{-5s}\right]$ [by integration by parts with $u = s$]

$\displaystyle = \lim_{t\to\infty}\left(-\tfrac{1}{5}te^{-5t} - \tfrac{1}{25}e^{-5t} + \tfrac{1}{25}\right) = 0 - 0 + \tfrac{1}{25}$ [by l'Hospital's Rule]

$= \tfrac{1}{25}.$ Convergent

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ [by substitution with $u = \ln x$, $du = dx/x$] $\displaystyle = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

30. $\displaystyle\int_6^8 \frac{4}{(x-6)^3}\,dx = \lim_{t\to6+}\int_t^8 4(x-6)^{-3}\,dx = \lim_{t\to6+}\left[-2(x-6)^{-2}\right]_t^8 = -2\lim_{t\to6+}\left[\frac{1}{2^2} - \frac{1}{(t-6)^2}\right] = \infty.$ Divergent

35. $I = \int_0^3 \frac{dx}{x^2 - 6x + 5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2 = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}$.

Now $\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \quad \Rightarrow \quad 1 = A(x-5) + B(x-1)$.

Set $x = 5$ to get $1 = 4B$, so $B = \frac{1}{4}$. Set $x = 1$ to get $1 = -4A$, so $A = -\frac{1}{4}$. Thus

$$\begin{aligned} I_1 &= \lim_{t \to 1^-} \int_0^t \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x-5} \right) dx = \lim_{t \to 1^-} \left[-\frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x-5| \right]_0^t \\ &= \lim_{t \to 1^-} \left[\left(-\tfrac{1}{4} \ln|t-1| + \tfrac{1}{4} \ln|t-5| \right) - \left(-\tfrac{1}{4} \ln|-1| + \tfrac{1}{4} \ln|-5| \right) \right] \\ &= \infty, \quad \text{since } \lim_{t \to 1^-} \left(-\tfrac{1}{4} \ln|t-1| \right) = \infty. \end{aligned}$$

Since I_1 is divergent, I is divergent.

36. $\int_{\pi/2}^{\pi} \csc x \, dx = \lim_{t \to \pi^-} \int_{\pi/2}^{t} \csc x \, dx = \lim_{t \to \pi^-} \left[\ln|\csc x - \cot x| \right]_{\pi/2}^{t} = \lim_{t \to \pi^-} \left[\ln(\csc t - \cot t) - \ln(1 - 0) \right]$

$$= \lim_{t \to \pi^-} \ln\left(\frac{1 - \cos t}{\sin t} \right) = \infty. \qquad \text{Divergent}$$

49. For $x > 0$, $\frac{x}{x^3 + 1} < \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2} \, dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\int_1^\infty \frac{x}{x^3 + 1} \, dx$ is convergent by the Comparison Theorem. $\int_0^1 \frac{x}{x^3 + 1} \, dx$ is a constant, so $\int_0^\infty \frac{x}{x^3 + 1} \, dx = \int_0^1 \frac{x}{x^3 + 1} \, dx + \int_1^\infty \frac{x}{x^3 + 1} \, dx$ is also convergent.

50. For $x \geq 1$, $\frac{2 + e^{-x}}{x} > \frac{2}{x}$ [since $e^{-x} > 0$] $> \frac{1}{x}$. $\int_1^\infty \frac{1}{x} \, dx$ is divergent by Equation 2 with $p = 1 \leq 1$, so $\int_1^\infty \frac{2 + e^{-x}}{x} \, dx$ is divergent by the Comparison Theorem.

54. For $0 < x \leq 1$, $\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$. Now

$I = \int_0^\pi \frac{1}{\sqrt{x}} \, dx = \lim_{t \to 0+} \int_t^\pi x^{-1/2} \, dx = \lim_{t \to 0+} \left[2x^{1/2} \right]_t^\pi = \lim_{t \to 0+} \left(2\pi - 2\sqrt{t} \right) = 2\pi - 0 = 2\pi$, so I is convergent, and by comparison, $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}} \, dx$ is convergent.

57. If $p = 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} [\ln x]_t^1 = \infty$. Divergent.

If $p \neq 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p < 0$]

$$= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\frac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\frac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\frac{1}{1-p}$.

58. Let $u = \ln x$. Then $du = dx/x \Rightarrow \int_e^\infty \frac{dx}{x\,(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\frac{1}{p-1}$ if $p > 1$ and diverges otherwise.

59. First suppose $p = -1$. Then

$\int_0^1 x^p \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \left[\tfrac{1}{2}(\ln x)^2\right]_t^1 = -\tfrac{1}{2}\lim_{t\to 0^+}(\ln t)^2 = -\infty$, so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$\int x^p \ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p < -1$, then $p + 1 < 0$, so

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p + 1 > 0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{H}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\frac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

61. (a) $I = \int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$, and $\int_0^\infty x\,dx = \lim_{t\to\infty}\int_0^t x\,dx = \lim_{t\to\infty}\left[\tfrac{1}{2}x^2\right]_0^t = \lim_{t\to\infty}\left[\tfrac{1}{2}t^2 - 0\right] = \infty$, so I is divergent.

(b) $\int_{-t}^t x\,dx = \left[\tfrac{1}{2}x^2\right]_{-t}^t = \tfrac{1}{2}t^2 - \tfrac{1}{2}t^2 = 0$, so $\lim_{t\to\infty}\int_{-t}^t x\,dx = 0$. Therefore, $\int_{-\infty}^\infty x\,dx \neq \lim_{t\to\infty}\int_{-t}^t x\,dx$.

62. Let $k = \dfrac{M}{2RT}$ so that $\overline{v} = \dfrac{4}{\sqrt{\pi}} k^{3/2} \displaystyle\int_0^\infty v^3 e^{-kv^2}\, dv$. Let I denote the integral and use parts to integrate I. Let $\alpha = v^2$,

$d\beta = ve^{-kv^2}\, dv \quad \Rightarrow \quad d\alpha = 2v\, dv$, $\beta = -\dfrac{1}{2k} e^{-kv^2}$:

$$I = \lim_{t\to\infty} \left[-\frac{1}{2k} v^2 e^{-kv^2} \right]_0^t + \frac{1}{k} \int_0^\infty ve^{-kv^2}\, dv_0^t = -\frac{1}{2k} \lim_{t\to\infty} \left(t^2 e^{-kt^2} \right) + \frac{1}{k} \lim_{t\to\infty} \left[-\frac{1}{2k} e^{-kv^2} \right]$$

$$\overset{\text{H}}{=} -\frac{1}{2k} \cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\overline{v} = \dfrac{4}{\sqrt{\pi}} k^{3/2} \cdot \dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

71. (a) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\, dt = \int_0^\infty e^{-st}\, dt = \lim_{n\to\infty} \left[-\frac{e^{-st}}{s} \right]_0^n = \lim_{n\to\infty} \left(\frac{e^{-sn}}{-s} + \frac{1}{s} \right)$. This converges to $\dfrac{1}{s}$ only if $s > 0$.

Therefore $F(s) = \dfrac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\, dt = \int_0^\infty e^t e^{-st}\, dt = \lim_{n\to\infty} \int_0^n e^{t(1-s)}\, dt = \lim_{n\to\infty} \left[\frac{1}{1-s} e^{t(1-s)} \right]_0^n$

$$= \lim_{n\to\infty} \left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s} \right)$$

This converges only if $1 - s < 0 \quad \Rightarrow \quad s > 1$, in which case $F(s) = \dfrac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\, dt = \lim\limits_{n\to\infty} \int_0^n te^{-st}\, dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\, dt \quad \Rightarrow \quad du = dt$,

$v = -\dfrac{e^{-st}}{s}$. Then $F(s) = \lim\limits_{n\to\infty} \left[-\dfrac{t}{s} e^{-st} - \dfrac{1}{s^2} e^{-st} \right]_0^n = \lim\limits_{n\to\infty} \left(\dfrac{-n}{se^{sn}} - \dfrac{1}{s^2 e^{sn}} + 0 + \dfrac{1}{s^2} \right) = \dfrac{1}{s^2}$ only if $s > 0$.

Therefore, $F(s) = \dfrac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.