2. (a) $\dfrac{x}{x^2+x-2} = \dfrac{x}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

(b) $\dfrac{x^2}{x^2+x+2} = \dfrac{(x^2+x+2)-(x+2)}{x^2+x+2} = 1 - \dfrac{x+2}{x^2+x+2}$

Notice that x^2+x+2 can't be factored because its discriminant is $b^2-4ac=-7<0$.

5. (a) $\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1 + \dfrac{1}{x^4-1}$ [or use long division] $= 1 + \dfrac{1}{(x^2-1)(x^2+1)}$

$$= 1 + \frac{1}{(x-1)(x+1)(x^2+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+4} + \dfrac{Et+F}{(t^2+4)^2}$

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}dx = \int \left(1 + \frac{6}{x-6}\right)dx = x + 6\ln|x-6| + C$

12. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2)+B(x+1)$. Substituting -2 for x gives $-3=-B \quad\Leftrightarrow\quad B=3$. Substituting -1 for x gives $-2=A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right)dx = \Big[-2\ln|x+1| + 3\ln|x+2|\Big]_0^1$$

$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \qquad \left[\text{or } \ln\tfrac{27}{32}\right]$$

16. $\dfrac{x^3-4x-10}{x^2-x-6} = x + 1 + \dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$. Then $3x-4 = A(x+2)+B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \quad\Leftrightarrow\quad A=1$ and $-10=-5B \quad\Leftrightarrow\quad B=2$, so

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 \left(x + 1 + \frac{1}{x-3} + \frac{2}{x+2}\right)dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2)\right]_0^1$$

$$= \left(\tfrac{1}{2} + 1 + \ln 2 + 2\ln 3\right) - (0 + 0 + \ln 3 + 2\ln 2) = \tfrac{3}{2} + \ln 3 - \ln 2 = \tfrac{3}{2} + \ln\tfrac{3}{2}$$

23. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2\,(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2)+B(x+2)+Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C \quad\Rightarrow\quad A=2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

34. $\dfrac{x^3}{x^3+1} = \dfrac{(x^3+1)-1}{x^3+1} = 1 - \dfrac{1}{x^3+1} = 1 - \left(\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}\right) \quad \Rightarrow \quad 1 = A(x^2-x+1) + (Bx+C)(x+1).$

Equate the terms of degree 2, 1 and 0 to get $0 = A + B$, $0 = -A + B + C$, $1 = A + C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\int \frac{x^3}{x^3+1}\,dx = \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}\right] dx = x - \tfrac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= x - \tfrac{1}{3}\ln|x+1| + \tfrac{1}{6}\ln(x^2-x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x-1)\right) + K$$

35. $\dfrac{1}{x(x^2+4)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2} \quad \Rightarrow \quad 1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$. Setting $x = 0$ gives $1 = 16A$, so $A = \frac{1}{16}$. Now compare coefficients.

$$1 = \tfrac{1}{16}(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2+4) + Dx^2 + Ex$$

$$1 = \tfrac{1}{16}x^4 + \tfrac{1}{2}x^2 + 1 + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex$$

$$1 = \left(\tfrac{1}{16} + B\right)x^4 + Cx^3 + \left(\tfrac{1}{2} + 4B + D\right)x^2 + (4C + E)x + 1$$

So $B + \frac{1}{16} = 0 \quad \Rightarrow \quad B = -\frac{1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \quad \Rightarrow \quad D = -\frac{1}{4}$, and $4C + E = 0 \quad \Rightarrow \quad E = 0$. Thus,

$$\int \frac{dx}{x(x^2+4)^2} = \int \left(\frac{\frac{1}{16}}{x} + \frac{-\frac{1}{16}x}{x^2+4} + \frac{-\frac{1}{4}x}{(x^2+4)^2}\right) dx = \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln|x^2+4| - \frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{x^2+4} + C$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C$$

41. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1 + \frac{4}{u^2-4}\right)du \qquad \text{[by long division]}$$

$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)} \quad (\star)$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A + B = 0$ and $-2A + 2B = 1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so $\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and $(\star)$ is

$$2 + 8\int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2}\right) du = 2 + 8\left[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\right]_3^4 = 2 + \Big[2\ln|u-2| - 2\ln|u+2|\Big]_3^4$$

$$= 2 + 2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4 = 2 + 2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2 + 2\ln\tfrac{2/6}{1/5}$$

$$= 2 + 2\ln\tfrac{5}{3} \;\text{ or }\; 2 + \ln\left(\tfrac{5}{3}\right)^2 = 2 + \ln\tfrac{25}{9}$$

44. Let $u=\sqrt{x}$. Then $x=u^2$, $dx=2u\,du$ $\Rightarrow$

$$\int_{1/3}^{3}\frac{\sqrt{x}}{x^2+x}\,dx=\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{u\cdot 2u\,du}{u^4+u^2}=2\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{du}{u^2+1}=2\left[\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}}=2\left(\tfrac{\pi}{3}-\tfrac{\pi}{6}\right)=\tfrac{\pi}{3}.$$

45. If we were to substitute $u=\sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u=\sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u=\sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u=\sqrt[6]{x}$. Then $x=u^6$, so $dx=6u^5\,du$ and $\sqrt{x}=u^3$, $\sqrt[3]{x}=u^2$. Thus,

$$\int\frac{dx}{\sqrt{x}-\sqrt[3]{x}}=\int\frac{6u^5\,du}{u^3-u^2}=6\int\frac{u^5}{u^2(u-1)}\,du=6\int\frac{u^3}{u-1}\,du$$

$$=6\int\left(u^2+u+1+\frac{1}{u-1}\right)du \qquad \text{[by long division]}$$

$$=6\left(\tfrac{1}{3}u^3+\tfrac{1}{2}u^2+u+\ln|u-1|\right)+C=2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

48. Let $u=\sin x$. Then $du=\cos x\,dx$ $\Rightarrow$

$$\int\frac{\cos x\,dx}{\sin^2 x+\sin x}=\int\frac{du}{u^2+u}=\int\frac{du}{u(u+1)}=\int\left[\frac{1}{u}-\frac{1}{u+1}\right]du=\ln\left|\frac{u}{u+1}\right|+C=\ln\left|\frac{\sin x}{1+\sin x}\right|+C.$$

51. Let $u=\ln(x^2-x+2)$, $dv=dx$. Then $du=\dfrac{2x-1}{x^2-x+2}\,dx$, $v=x$, and (by integration by parts)

$$\int\ln(x^2-x+2)\,dx=x\ln(x^2-x+2)-\int\frac{2x^2-x}{x^2-x+2}\,dx=x\ln(x^2-x+2)-\int\left(2+\frac{x-4}{x^2-x+2}\right)dx$$

$$=x\ln(x^2-x+2)-2x-\int\frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx+\frac{7}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{7}{4}}$$

$$=x\ln(x^2-x+2)-2x-\frac{1}{2}\ln(x^2-x+2)+\frac{7}{2}\int\frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x-\frac{1}{2}=\frac{\sqrt{7}}{2}u,\\ dx=\frac{\sqrt{7}}{2}\,du,\\ (x-\frac{1}{2})^2+\frac{7}{4}=\frac{7}{4}(u^2+1)\end{array}\right]$$

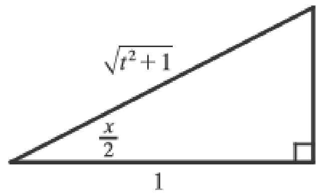
$$=(x-\tfrac{1}{2})\ln(x^2-x+2)-2x+\sqrt{7}\tan^{-1}u+C$$

$$=(x-\tfrac{1}{2})\ln(x^2-x+2)-2x+\sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}}+C$$

55. $$\int\frac{dx}{x^2-2x}=\int\frac{dx}{(x-1)^2-1}=\int\frac{du}{u^2-1} \qquad [\text{put } u=x-1]$$

$$=\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+C \quad [\text{by Equation 6}] \quad =\frac{1}{2}\ln\left|\frac{x-2}{x}\right|+C$$

57. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives



$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$

(b) $\cos x = \cos\left(2 \cdot \frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$$= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

(c) $\frac{x}{2} = \arctan t \quad \Rightarrow \quad x = 2\arctan t \quad \Rightarrow \quad dx = \frac{2}{1+t^2}\, dt$

58. Let $t = \tan(x/2)$. Then, using Exercise 57, $dx = \frac{2}{1+t^2}\, dt$, $\sin x = \frac{2t}{1+t^2} \quad \Rightarrow$

$$\int \frac{dx}{3 - 5\sin x} = \int \frac{2\, dt/(1+t^2)}{3 - 10t/(1+t^2)} = \int \frac{2\, dt}{3(1+t^2) - 10t} = 2\int \frac{dt}{3t^2 - 10t + 3}$$

$$= \frac{1}{4}\int \left[\frac{1}{t-3} - \frac{3}{3t-1}\right] dt = \tfrac{1}{4}(\ln|t-3| - \ln|3t-1|) + C = \frac{1}{4}\ln\left|\frac{\tan(x/2) - 3}{3\tan(x/2) - 1}\right| + C$$

65. $\frac{P+S}{P\left[(r-1)P - S\right]} = \frac{A}{P} + \frac{B}{(r-1)P - S} \quad \Rightarrow \quad P + S = A\left[(r-1)P - S\right] + BP = \left[(r-1)A + B\right]P - AS \quad \Rightarrow$

$(r-1)A + B = 1, -A = 1 \quad \Rightarrow \quad A = -1, B = r$. Now

$$t = \int \frac{P+S}{P\left[(r-1)P - S\right]}\, dP = \int \left[\frac{-1}{P} + \frac{r}{(r-1)P - S}\right] dP = -\int \frac{dP}{P} + \frac{r}{r-1}\int \frac{r-1}{(r-1)P - S}\, dP$$

so $t = -\ln P + \frac{r}{r-1}\ln|(r-1)P - S| + C$. Here $r = 0.10$ and $S = 900$, so

$t = -\ln P + \frac{0.1}{-0.9}\ln|-0.9P - 900| + C = -\ln P - \frac{1}{9}\ln(|-1|\,|0.9P + 900|) = -\ln P - \frac{1}{9}\ln(0.9P + 900) + C.$

When $t = 0$, $P = 10{,}000$, so $0 = -\ln 10{,}000 - \frac{1}{9}\ln(9900) + C$. Thus, $C = \ln 10{,}000 + \frac{1}{9}\ln 9900$ [≈ 10.2326], so our equation becomes

$$t = \ln 10{,}000 - \ln P + \tfrac{1}{9}\ln 9900 - \tfrac{1}{9}\ln(0.9P + 900) = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{9900}{0.9P + 900}$$

$$= \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{1100}{0.1P + 100} = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{11{,}000}{P + 1000}$$