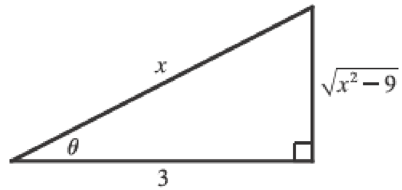
1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\,\tan\theta\,d\theta$ and



$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}$$
$$= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta}\,3\sec\theta\,\tan\theta\,d\theta = \tfrac{1}{9}\int \cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta+\pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

4. Let $x = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta\,d\theta$ and $\sqrt{16-x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\,|\cos\theta| = 4\cos\theta$. When $x = 0$, $4\sin\theta = 0 \;\Rightarrow\; \theta = 0$, and when $x = 2\sqrt{3}$, $4\sin\theta = 2\sqrt{3} \;\Rightarrow\; \sin\theta = \frac{\sqrt{3}}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta}\,4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta = 4^3\int_0^{\pi/3}(1-\cos^2\theta)\,\sin\theta\,d\theta$$
$$\overset{c}{=} -4^3\int_1^{1/2}(1-u^2)\,du = -64\left[u - \tfrac{1}{3}u^3\right]_1^{1/2}$$
$$= -64\left[\left(\tfrac{1}{2} - \tfrac{1}{24}\right) - \left(1 - \tfrac{1}{3}\right)\right] = -64\left(-\tfrac{5}{24}\right) = \tfrac{40}{3}$$

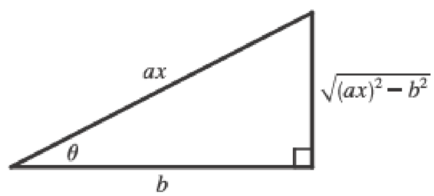
Or: Let $u = 16 - x^2$, $x^2 = 16 - u$, $du = -2x\,dx$.

12. $\int_0^1 x\sqrt{x^2+4}\,dx = \int_4^5 \sqrt{u}\left(\tfrac{1}{2}\,du\right) \quad [u = x^2+4,\ du = 2x\,dx] \quad = \tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_4^5 = \tfrac{1}{3}(5\sqrt{5} - 8)$

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \;\Rightarrow$

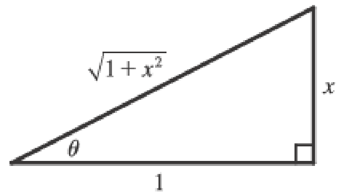
$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$.

So $\sqrt{(ax)^2 - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\,\tan\theta\,d\theta$, and



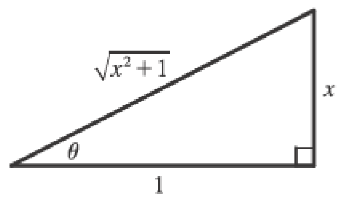
$$\int \frac{dx}{[(ax)^2 - b^2]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\,\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\,\cot\theta\,d\theta$$
$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2 - b^2}} + C$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



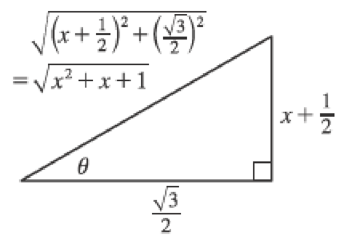
$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$
$$= \int(\csc\theta + \sec\theta\,\tan\theta)\,d\theta$$
$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.39]}$$
$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$$

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so



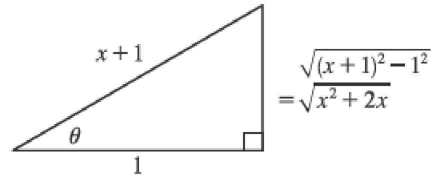
$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4} \sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$
$$= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{[by Example 7.2.8]}$$
$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln(1+\sqrt{2}) - 0 - \ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln(1+\sqrt{2})\right]$$

25. $x^2 + x + 1 = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4} = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$. Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, so $dx = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\sec\theta$. Then



$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$
$$= \int\left(\tfrac{\sqrt{3}}{2}\tan\theta - \tfrac{1}{2}\right)\sec\theta\,d\theta = \int \tfrac{\sqrt{3}}{2}\tan\theta\,\sec\theta\,d\theta - \int \tfrac{1}{2}\sec\theta\,d\theta$$
$$= \tfrac{\sqrt{3}}{2}\sec\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C_1$$
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\sqrt{x^2+x+1} + \tfrac{2}{\sqrt{3}}\left(x+\tfrac{1}{2}\right)\right| + C_1$$
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}}\left[\sqrt{x^2+x+1} + \left(x+\tfrac{1}{2}\right)\right]\right| + C_1$$
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C_1$$
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln\left(\sqrt{x^2+x+1} + x + \tfrac{1}{2}\right) + C, \quad \text{where } C = C_1 - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}}$$

27. $x^2+2x=(x^2+2x+1)-1=(x+1)^2-1$. Let $x+1=1\sec\theta$, so $dx=\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2+2x}=\tan\theta$. Then

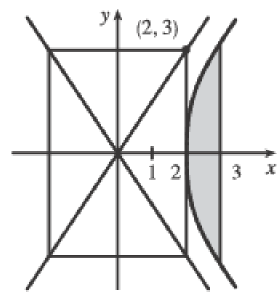


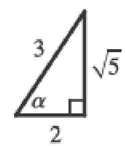
$$\begin{aligned}\int\sqrt{x^2+2x}\,dx&=\int\tan\theta\,(\sec\theta\,\tan\theta\,d\theta)=\int\tan^2\theta\,\sec\theta\,d\theta\\&=\int(\sec^2\theta-1)\,\sec\theta\,d\theta=\int\sec^3\theta\,d\theta-\int\sec\theta\,d\theta\\&=\tfrac12\sec\theta\,\tan\theta+\tfrac12\ln|\sec\theta+\tan\theta|-\ln|\sec\theta+\tan\theta|+C\\&=\tfrac12\sec\theta\,\tan\theta-\tfrac12\ln|\sec\theta+\tan\theta|+C=\tfrac12(x+1)\sqrt{x^2+2x}-\tfrac12\ln\left|x+1+\sqrt{x^2+2x}\right|+C\end{aligned}$$

34. $9x^2-4y^2=36\quad\Rightarrow\quad y=\pm\frac32\sqrt{x^2-4}\quad\Rightarrow$

area $=2\int_2^3\frac32\sqrt{x^2-4}\,dx=3\int_2^3\sqrt{x^2-4}\,dx$

$$\begin{aligned}&=3\int_0^\alpha 2\tan\theta\,2\sec\theta\,\tan\theta\,d\theta\qquad\left[\begin{array}{c}\text{where } x=2\sec\theta,\\ dx=2\sec\theta\,\tan\theta\,d\theta,\\ \alpha=\sec^{-1}\left(\frac32\right)\end{array}\right]\\&=12\int_0^\alpha(\sec^2\theta-1)\sec\theta\,d\theta=12\int_0^\alpha(\sec^3\theta-\sec\theta)\,d\theta\\&=12\left[\tfrac12(\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|)-\ln|\sec\theta+\tan\theta|\right]_0^\alpha\\&=6\Big[\sec\theta\,\tan\theta-\ln|\sec\theta+\tan\theta|\Big]_0^\alpha\\&=6\left[\frac{3\sqrt5}{4}-\ln\left(\frac32+\frac{\sqrt5}{2}\right)\right]=\frac{9\sqrt5}{2}-6\ln\left(\frac{3+\sqrt5}{2}\right)\end{aligned}$$





35. Area of $\triangle POQ=\frac12(r\cos\theta)(r\sin\theta)=\frac12r^2\sin\theta\,\cos\theta$. Area of region $PQR=\int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$.

Let $x=r\cos u\quad\Rightarrow\quad dx=-r\sin u\,du$ for $\theta\le u\le\frac{\pi}{2}$. Then we obtain

$$\begin{aligned}\int\sqrt{r^2-x^2}\,dx&=\int r\sin u\,(-r\sin u)\,du=-r^2\int\sin^2u\,du=-\tfrac12r^2(u-\sin u\,\cos u)+C\\&=-\tfrac12r^2\cos^{-1}(x/r)+\tfrac12x\sqrt{r^2-x^2}+C\end{aligned}$$

so

$$\begin{aligned}\text{area of region } PQR&=\tfrac12\left[-r^2\cos^{-1}(x/r)+x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r}\\&=\tfrac12\left[0-(-r^2\theta+r\cos\theta\,r\sin\theta)\right]=\tfrac12r^2\theta-\tfrac12r^2\sin\theta\,\cos\theta\end{aligned}$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $=\frac12r^2\theta$.

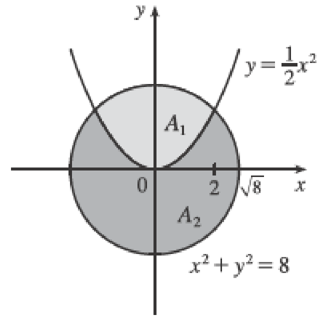
40. The curves intersect when $x^2 + \left(\frac{1}{2}x^2\right)^2 = 8 \quad \Leftrightarrow \quad x^2 + \frac{1}{4}x^4 = 8 \quad \Leftrightarrow \quad x^4 + 4x^2 - 32 = 0 \quad \Leftrightarrow$

$(x^2 + 8)(x^2 - 4) = 0 \quad \Leftrightarrow \quad x = \pm 2$. The area inside the circle and above the parabola is given by

$$A_1 = \int_{-2}^{2} \left(\sqrt{8 - x^2} - \tfrac{1}{2}x^2\right) dx = 2\int_0^2 \sqrt{8 - x^2}\, dx - 2\int_0^2 \tfrac{1}{2}x^2\, dx$$

$$= 2\left[\tfrac{1}{2}(8)\sin^{-1}\left(\tfrac{2}{\sqrt{8}}\right) + \tfrac{1}{2}(2)\sqrt{8 - 2^2} - \tfrac{1}{2}\left[\tfrac{1}{3}x^3\right]_0^2\right] \qquad \text{[by Exercise 39]}$$

$$= 8\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) + 2\sqrt{4} - \tfrac{8}{3} = 8\left(\tfrac{\pi}{4}\right) + 4 - \tfrac{8}{3} = 2\pi + \tfrac{4}{3}$$



Since the area of the disk is $\pi\left(\sqrt{8}\right)^2 = 8\pi$, the area inside the circle and below the parabola ia $A_2 = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}$.