1. Let $u = \ln x$, $dv = x^2\,dx \Rightarrow du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$$\int x^2 \ln x\,dx = (\ln x)\left(\tfrac{1}{3}x^3\right) - \int \left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\right)\,dx = \tfrac{1}{3}x^3\ \ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{1}{3}x^3\ \ln x - \tfrac{1}{3}\left(\tfrac{1}{3}x^3\right) + C$$
$$= \tfrac{1}{3}x^3\ \ln x - \tfrac{1}{9}x^3 + C \quad \left[\text{or } \tfrac{1}{3}x^3\left(\ln x - \tfrac{1}{3}\right) + C\right]$$

6. Let $u = t$, $dv = \sin 2t\,dt \Rightarrow du = dt$, $v = -\frac{1}{2}\cos 2t$. Then

$$\int t\sin 2t\,dt = -\tfrac{1}{2}t\cos 2t + \tfrac{1}{2}\int \cos 2t\,dt = -\tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t + C.$$

9. Let $u = \ln(2x+1)$, $dv = dx \Rightarrow du = \dfrac{2}{2x+1}\,dx$, $v = x$. Then

$$\int \ln(2x+1)\,dx = x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}\,dx$$
$$= x\ln(2x+1) - \int \left(1 - \frac{1}{2x+1}\right)dx = x\ln(2x+1) - x + \tfrac{1}{2}\ln(2x+1) + C$$
$$= \tfrac{1}{2}(2x+1)\ \ln(2x+1) - x + C$$

15. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,
$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \Rightarrow$
$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,
$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

20. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),
$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$.
Next let $U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,
$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So
$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1} + 1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

23. Let $u = \ln x$, $dv = x^{-2}\,dx \Rightarrow du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

29. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \Rightarrow du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then
$I = \int \cos x\ln(\sin x)\,dx = \sin x\ln(\sin x) - \int \cos x\,dx = \sin x\ln(\sin x) - \sin x + C$.
Another method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int \ln t\,dt = t\ln t - t + C$ (see Example 2) and so
$I = \sin x\,(\ln\sin x - 1) + C$.

34. Let $x = -t^2$, so that $dx = -2t\,dt$. Thus, $\int t^3 e^{-t^2}\,dt = \int (-t^2)e^{-t^2}\left(\frac{1}{2}\right)(-2t\,dt) = \frac{1}{2}\int xe^x\,dx$. Now use parts with $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$\frac{1}{2}\int xe^x\,dx = \frac{1}{2}\left(xe^x - \int e^x\,dx\right) = \frac{1}{2}xe^x - \frac{1}{2}e^x + C = -\frac{1}{2}(1-x)e^x + C = -\frac{1}{2}(1+t^2)e^{-t^2} + C.$

38. Let $y = \ln x$, so that $dy = \frac{1}{x}\,dx \quad\Rightarrow\quad dx = x\,dy = e^y\,dy$. Thus,

$\int \sin(\ln x)\,dx = \int \sin y\, e^y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C \quad$ [by Example 4] $\quad = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C.$

43. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C.$

47. Let $u = (\ln x)^n$, $dv = dx \quad\Rightarrow\quad du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int (\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$

48. Let $u = x^n$, $dv = e^x\,dx \quad\Rightarrow\quad du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

52. By repeated applications of the reduction formula in Exercise 48,

$$\begin{aligned}\int x^4 e^x\,dx &= x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left(x^3e^x - 3\int x^2e^x\,dx\right)\\ &= x^4e^x - 4x^3e^x + 12\left(x^2e^x - 2\int x^1e^x\,dx\right) = x^4e^x - 4x^3e^x + 12x^2e^x - 24\left(x^1e^x - \int x^0e^x\,dx\right)\\ &= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C \quad \left[\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C\right]\end{aligned}$$

62. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$\begin{aligned}H &= \int_0^{60}\left[-gt - v_e\ln\left(\frac{m-rt}{m}\right)\right]dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60}\ln(m-rt)\,dt - \int_0^{60}\ln m\,dt\right]\\ &= -g(1800) + v_e(\ln m)(60) - v_e\int_0^{60}\ln(m-rt)\,dt\end{aligned}$$

Let $u = \ln(m - rt)$, $dv = dt \quad\Rightarrow\quad du = \dfrac{1}{m-rt}(-r)\,dt$, $v = t$. Then

$$\begin{aligned}\int_0^{60}\ln(m-rt)\,dt &= \Big[t\ln(m-rt)\Big]_0^{60} + \int_0^{60}\frac{rt}{m-rt}\,dt = 60\ln(m-60r) + \int_0^{60}\left(-1 + \frac{m}{m-rt}\right)dt\\ &= 60\ln(m-60r) + \left[-t - \frac{m}{r}\ln(m-rt)\right]_0^{60} = 60\ln(m-60r) - 60 - \frac{m}{r}\ln(m-60r) + \frac{m}{r}\ln m\end{aligned}$$

So $H = -1800g + 60v_e\ln m - 60v_e\ln(m-60r) + 60v_e + \dfrac{m}{r}v_e\ln(m-60r) - \dfrac{m}{r}v_e\ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

63. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2 e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \;\Rightarrow\; du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2 e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \;\Rightarrow\; dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right)$$

$$= -t^2e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}$$

65. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \;\Rightarrow\; du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7-2) = 12 - 5 - 5 = 2.$

We used the fact that f'' is continuous to guarantee that I exists.