1. If $f(x)=\sum\limits_{n=0}^{\infty}c_nx^n$ has radius of convergence 10, then $f'(x)=\sum\limits_{n=1}^{\infty}nc_nx^{n-1}$ also has radius of convergence 10 by Theorem 2.

3. Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x)=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum\limits_{n=0}^{\infty}(-x)^n=\sum\limits_{n=0}^{\infty}(-1)^nx^n$ with $|-x|<1 \quad\Leftrightarrow\quad |x|<1$, so $R=1$ and $I=(-1,1)$.

6. $f(x)=\frac{1}{x+10}=\frac{1}{10}\left(\frac{1}{1-(-x/10)}\right)=\frac{1}{10}\sum\limits_{n=0}^{\infty}\left(-\frac{x}{10}\right)^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{10^{n+1}}x^n$. The series converges when $\left|\frac{x}{10}\right|<1$, that is, when $|x|<10$, so $R=10$ and $I=(-10,10)$.

8. $f(x)=\frac{x}{2x^2+1}=x\left(\frac{1}{1-(-2x^2)}\right)=x\sum\limits_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum\limits_{n=0}^{\infty}(-1)^n2^nx^{2n+1}$. The series converges when $|-2x^2|<1 \quad\Rightarrow\quad |x^2|<\frac{1}{2} \quad\Rightarrow\quad |x|<\frac{1}{\sqrt{2}}$, so $R=\frac{1}{\sqrt{2}}$ and $I=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

11. $f(x)=\frac{3}{x^2-x-2}=\frac{3}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1} \quad\Rightarrow\quad 3=A(x+1)+B(x-2)$. Let $x=2$ to get $A=1$ and $x=-1$ to get $B=-1$. Thus

$$\frac{3}{x^2-x-2}=\frac{1}{x-2}-\frac{1}{x+1}=\frac{1}{-2}\left(\frac{1}{1-(x/2)}\right)-\frac{1}{1-(-x)}=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n-\sum_{n=0}^{\infty}(-x)^n$$

$$=\sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{1}{2}\right)^n-1(-1)^n\right]x^n=\sum_{n=0}^{\infty}\left[(-1)^{n+1}-\frac{1}{2^{n+1}}\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x\in(-2,2)$ and the second converges for $(-1,1)$. Thus, the sum converges for $x\in(-1,1)=I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n\, x^n\right]$ [from Exercise 3]

$= \sum\limits_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$ [from Theorem 2(i)] $= \sum\limits_{n=0}^{\infty}(-1)^n (n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n (n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum\limits_{n=1}^{\infty}(-1)^n (n+1)n x^{n-1} = \frac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n (n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n (n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n (n+2)(n+1)x^{n+2}$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum\limits_{n=2}^{\infty}(-1)^n (n)(n-1)x^n$ with $R = 1$.

14. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$f(x) = \ln(1+x) = \int \dfrac{dx}{1+x} = \int\left[\sum\limits_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum\limits_{n=0}^{\infty}(-1)^n \dfrac{x^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}$

[$C = 0$ since $f(0) = \ln 1 = 0$], with $R = 1$

(b) $f(x) = x\ln(1+x) = x\left[\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum\limits_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$ with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

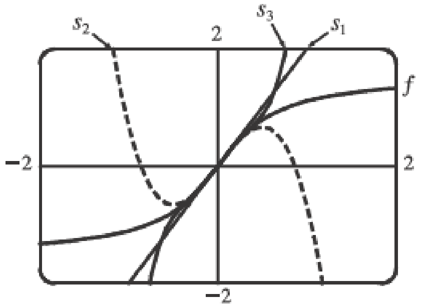
15. $f(x) = \ln(5-x) = -\displaystyle\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5} = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n\,5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \quad\Leftrightarrow\quad |x| < 5$, so $R = 5$.

22. $f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n \, dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} \, dx$

$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \qquad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0]$$

The series converges when $\left|4x^2\right| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and $f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test. The partial sums are $s_1 = \frac{2x}{1}$, $s_2 = s_1 - \frac{2^3 x^3}{3}$, $s_3 = s_2 + \frac{2^5 x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for $|t| < 1$, so $\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$ and $\int \frac{\ln(1-t)}{t}\,dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$.

By Theorem 2, $R = 1$.

29. We substitute $3x$ for x in Example 7, and find that

$$\int x\arctan(3x)\,dx = \int x \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+2}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+3}}{(2n+1)(2n+3)}$$

So
$$\int_0^{0.1} x\arctan(3x)\,dx = \left[\frac{3x^3}{1\cdot 3} - \frac{3^3 x^5}{3\cdot 5} + \frac{3^5 x^7}{5\cdot 7} - \frac{3^7 x^9}{7\cdot 9} + \cdots\right]_0^{0.1}$$
$$= \frac{1}{10^3} - \frac{9}{5\times 10^5} + \frac{243}{35\times 10^7} - \frac{2187}{63\times 10^9} + \cdots.$$

The series is alternating, so if we use three terms, the error is at most $\frac{2187}{63\times 10^9} \approx 3.5\times 10^{-8}$. So

$$\int_0^{0.1} x\arctan(3x)\,dx \approx \frac{1}{10^3} - \frac{9}{5\times 10^5} + \frac{243}{35\times 10^7} \approx 0.000\,983 \text{ to six decimal places.}$$

30. $$\int_0^{0.3}\frac{x^2}{1+x^4}\,dx=\int_0^{0.3}x^2\sum_{n=0}^{\infty}(-1)^n x^{4n}\,dx=\sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{4n+3}}{4n+3}\right]_0^{0.3}=\sum_{n=0}^{\infty}\frac{(-1)^n\,3^{4n+3}}{(4n+3)10^{4n+3}}$$

$$=\frac{3^3}{3\times10^3}-\frac{3^7}{7\times10^7}+\frac{3^{11}}{11\times10^{11}}-\cdots$$

The series is alternating, so if we use only two terms, the error is at most $\frac{3^{11}}{11\times10^{11}}\approx 0.000\,000\,16$. So, to six decimal places, $\int_0^{0.3}\frac{x^2}{1+x^4}\,dx\approx\frac{3^3}{3\times10^3}-\frac{3^7}{7\times10^7}\approx 0.008\,969$.

33. (a) $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\,2nx^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\,2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$\begin{aligned}x^2J_0''(x)+xJ_0'(x)+x^2J_0(x)&=\sum_{n=1}^{\infty}\frac{(-1)^n\,2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\,2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}\\&=\sum_{n=1}^{\infty}\frac{(-1)^n\,2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\,2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{2^{2n-2}\left[(n-1)!\right]^2}\\&=\sum_{n=1}^{\infty}\frac{(-1)^n\,2n(2n-1)x^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n\,2nx^{2n}}{2^{2n}(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^{-1}2^2n^2x^{2n}}{2^{2n}(n!)^2}\\&=\sum_{n=1}^{\infty}(-1)^n\left[\frac{2n(2n-1)+2n-2^2n^2}{2^{2n}(n!)^2}\right]x^{2n}\\&=\sum_{n=1}^{\infty}(-1)^n\left[\frac{4n^2-2n+2n-4n^2}{2^{2n}(n!)^2}\right]x^{2n}=0\end{aligned}$$

(b) $$\int_0^1 J_0(x)\,dx=\int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right]dx=\int_0^1\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\cdots\right)dx$$

$$=\left[x-\frac{x^3}{3\cdot4}+\frac{x^5}{5\cdot64}-\frac{x^7}{7\cdot2304}+\cdots\right]_0^1=1-\frac{1}{12}+\frac{1}{320}-\frac{1}{16{,}128}+\cdots$$

Since $\frac{1}{16{,}128}\approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places,

$\int_0^1 J_0(x)\,dx\approx 1-\frac{1}{12}+\frac{1}{320}\approx 0.920$.