

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

- (i) 0 if the series converges only when $x = a$
- (ii) ∞ if the series converges for all x , or
- (iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

$$3. \text{ If } a_n = \frac{x^n}{\sqrt{n}}, \text{ then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{\sqrt{n+1}/\sqrt{n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{\sqrt{1+1/n}} = |x|.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the

endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because it is a p -series with $p = \frac{1}{2} \leq 1$. When $x = -1$,

the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-1, 1)$.

$$7. \text{ If } a_n = \frac{x^n}{n!}, \text{ then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1 \text{ for all real } x.$$

So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

10. If $a_n = \frac{10^n x^n}{n^3}$, then

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{10^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{10^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{10x n^3}{(n+1)^3} \right| = \lim_{n \rightarrow \infty} \frac{10|x|}{(1+1/n)^3} = \frac{10|x|}{1^3} = 10|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ converges when $10|x| < 1 \Leftrightarrow |x| < \frac{1}{10}$, so the radius of convergence is $R = \frac{1}{10}$.

When $x = -\frac{1}{10}$, the series converges by the Alternating Series Test; when $x = \frac{1}{10}$, the series converges because it is a p -series with $p = 3 > 1$. Thus, the interval of convergence is $I = \left[-\frac{1}{10}, \frac{1}{10}\right]$.

$$13. \text{ If } a_n = (-1)^n \frac{x^n}{4^n \ln n}, \text{ then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{4^{n+1} \ln(n+1)} \cdot \frac{4^n \ln n}{x^n} \right| = \frac{|x|}{4} \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} \cdot 1$$

[by l'Hospital's Rule] $= \frac{|x|}{4}$. By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \Leftrightarrow |x| < 4$, so $R = 4$. When

$$x = -4, \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}. \text{ Since } \ln n < n \text{ for } n \geq 2, \frac{1}{\ln n} > \frac{1}{n} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n} \text{ is the}$$

divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$,

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}, \text{ which converges by the Alternating Series Test. Thus, } I = (-4, 4].$$

$$15. \text{ If } a_n = \frac{(x-2)^n}{n^2+1}, \text{ then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| = |x-2| \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} = |x-2|. \text{ By the}$$

Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ converges when $|x-2| < 1$ [$R = 1$] $\Leftrightarrow -1 < x-2 < 1 \Leftrightarrow 1 < x < 3$. When

$$x = 1, \text{ the series } \sum_{n=0}^{\infty} (-1)^n \frac{1}{n^2+1} \text{ converges by the Alternating Series Test; when } x = 3, \text{ the series } \sum_{n=0}^{\infty} \frac{1}{n^2+1} \text{ converges by}$$

comparison with the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ [$p = 2 > 1$]. Thus, the interval of convergence is $I = [1, 3]$.

$$22. a_n = \frac{n(x-4)^n}{n^3+1}, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)|x-4|^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{n|x-4|^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \frac{n^3+1}{n^3+3n^2+3n+2} |x-4| = |x-4|.$$

By the Ratio Test, the series converges when $|x-4| < 1$ [so $R = 1$] $\Leftrightarrow -1 < x-4 < 1 \Leftrightarrow 3 < x < 5$. When

$$|x-4| = 1, \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}, \text{ which converges by comparison with the convergent } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ [} p = 2 > 1\text{].}$$

Thus, $I = [3, 5]$.

$$24. a_n = \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{n^2 x^n}{2^n n!} = \frac{n x^n}{2^n (n-1)!}, \text{ so}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)|x|^{n+1}}{2^{n+1} n!} \cdot \frac{2^n (n-1)!}{n|x|^n} = \lim_{n \rightarrow \infty} \frac{n+1}{n^2} \frac{|x|}{2} = 0. \text{ Thus, by the Ratio Test, the series converges for}$$

all real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \leq 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n (-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n / (n4^n)$.]

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n (-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.

33. No. If a power series is centered at a , its interval of convergence is symmetric about a . If a power series has an infinite radius of convergence, then its interval of convergence must be $(-\infty, \infty)$, not $[0, \infty)$.