

2. The series  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  has positive terms and  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left[ \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right] = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$ , so the series is absolutely convergent by the Ratio Test.

5.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[n]{n}}$  converges by the Alternating Series Test, but  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$  is a divergent  $p$ -series ( $p = \frac{1}{4} \leq 1$ ), so the given series is conditionally convergent.

7.  $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left[ \frac{(k+1) \left(\frac{2}{3}\right)^{k+1}}{k \left(\frac{2}{3}\right)^k} \right] = \lim_{k \rightarrow \infty} \frac{k+1}{k} \left(\frac{2}{3}\right)^1 = \frac{2}{3} \lim_{k \rightarrow \infty} \left( 1 + \frac{1}{k} \right) = \frac{2}{3}(1) = \frac{2}{3} < 1$ , so the series

$\sum_{n=1}^{\infty} k \left(\frac{2}{3}\right)^k$  is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

8.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[ \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} \right] = \lim_{n \rightarrow \infty} \frac{n+1}{100} = \infty$ , so the series  $\sum_{n=1}^{\infty} \frac{n!}{100^n}$  diverges by the Ratio Test.

9.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[ \frac{(1.1)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(1.1)^n} \right] = \lim_{n \rightarrow \infty} \frac{(1.1)n^4}{(n+1)^4} = (1.1) \lim_{n \rightarrow \infty} \frac{1}{\frac{(n+1)^4}{n^4}} = (1.1) \lim_{n \rightarrow \infty} \frac{1}{(1+1/n)^4}$   
 $= (1.1)(1) = 1.1 > 1$ ,

so the series  $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$  diverges by the Ratio Test.

11. Since  $0 \leq \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3} = e \left( \frac{1}{n^3} \right)$  and  $\sum_{n=1}^{\infty} \frac{1}{n^3}$  is a convergent  $p$ -series [ $p = 3 > 1$ ],  $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$  converges, and so

$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$  is absolutely convergent.

12.  $\left| \frac{\sin 4n}{4^n} \right| \leq \frac{1}{4^n}$ , so  $\sum_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right|$  converges by comparison with the convergent geometric series  $\sum_{n=1}^{\infty} \frac{1}{4^n}$  [ $|r| = \frac{1}{4} < 1$ ].

Thus,  $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$  is absolutely convergent.

17.  $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$  converges by the Alternating Series Test since  $\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$  and  $\left\{ \frac{1}{\ln n} \right\}$  is decreasing. Now  $\ln n < n$ , so

$\frac{1}{\ln n} > \frac{1}{n}$ , and since  $\sum_{n=2}^{\infty} \frac{1}{n}$  is the divergent (partial) harmonic series,  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$  diverges by the Comparison Test. Thus,

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$  is conditionally convergent.

21.  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{2 + 1/n^2} = \frac{1}{2} < 1$ , so the series  $\sum_{n=1}^{\infty} \left( \frac{n^2 + 1}{2n^2 + 1} \right)^n$  is absolutely convergent by the Root Test.

29. By the recursive definition,  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5n + 1}{4n + 3} \right| = \frac{5}{4} > 1$ , so the series diverges by the Ratio Test.

30. By the recursive definition,  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 + \cos n}{\sqrt{n}} \right| = 0 < 1$ , so the series converges absolutely by the Ratio Test.

$$\begin{aligned} 38. (a) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{[4(n+1)]! [1103 + 26,390(n+1)]}{[(n+1)!]^4 396^{4(n+1)}} \cdot \frac{(n!)^4 396^{4n}}{(4n)! (1103 + 26,390n)} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(26,390n + 27,493)}{(n+1)^4 396^4 (26,390n + 1103)} = \frac{4^4}{396^4} = \frac{1}{99^4} < 1, \end{aligned}$$

so by the Ratio Test, the series  $\sum_{n=0}^{\infty} \frac{(4n)! (1103 + 26,390n)}{(n!)^4 396^{4n}}$  converges.

$$(b) \frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)! (1103 + 26,390n)}{(n!)^4 396^{4n}}$$

With the first term ( $n = 0$ ),  $\frac{1}{\pi} \approx \frac{2\sqrt{2}}{9801} \cdot \frac{1103}{1} \Rightarrow \pi \approx 3.141\,592\,73$ , so we get 6 correct decimal places of  $\pi$ , which is 3.141 592 653 589 793 238 to 18 decimal places.

With the second term ( $n = 1$ ),  $\frac{1}{\pi} \approx \frac{2\sqrt{2}}{9801} \left( \frac{1103}{1} + \frac{4! (1103 + 26,390)}{396^4} \right) \Rightarrow \pi \approx 3.141\,592\,653\,589\,793\,878$ , so

we get 15 correct decimal places of  $\pi$ .