

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

(b) If $a_n < b_n$ for all n , then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all n , then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

4. $\frac{n^3}{n^4 - 1} > \frac{n^3}{n^4} = \frac{1}{n}$ for all $n \geq 2$, so $\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 1}$ diverges by comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges because it is a p -series with $p = 1 \leq 1$ (the harmonic series).

7. $\frac{9^n}{3 + 10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$ for all $n \geq 1$. $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series ($|r| = \frac{9}{10} < 1$), so $\sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n}$ converges by the Comparison Test.

9. $\frac{\cos^2 n}{n^2 + 1} \leq \frac{1}{n^2 + 1} < \frac{1}{n^2}$, so the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2 + 1}$ converges by comparison with the p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ [$p = 2 > 1$].

14. $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p -series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ [$p = \frac{1}{2} \leq 1$].

18. Use the Limit Comparison Test with $a_n = \frac{1}{2n+3}$ and $b_n = \frac{1}{n}$: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{2n+3} = \lim_{n \rightarrow \infty} \frac{1}{2 + (3/n)} = \frac{1}{2} > 0$.

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{2n+3}$.

19. Use the Limit Comparison Test with $a_n = \frac{1+4^n}{1+3^n}$ and $b_n = \frac{4^n}{3^n}$:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1+4^n}{1+3^n}}{\frac{4^n}{3^n}} = \lim_{n \rightarrow \infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n} = \lim_{n \rightarrow \infty} \frac{1+4^n}{4^n} \cdot \frac{3^n}{1+3^n} = \lim_{n \rightarrow \infty} \left(\frac{1}{4^n} + 1\right) \cdot \frac{1}{\frac{1}{3^n} + 1} = 1 > 0$$

Since the geometric series $\sum b_n = \sum \left(\frac{4}{3}\right)^n$ diverges, so does $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$. Alternatively, use the Comparison Test with

$$\frac{1+4^n}{1+3^n} > \frac{1+4^n}{3^n+3^n} > \frac{4^n}{2(3^n)} = \frac{1}{2} \left(\frac{4}{3}\right)^n \text{ or use the Test for Divergence.}$$

24. If $a_n = \frac{n^2 - 5n}{n^3 + n + 1}$ and $b_n = \frac{1}{n}$, then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^3 - 5n^2}{n^3 + n + 1} = \lim_{n \rightarrow \infty} \frac{1 - 5/n}{1 + 1/n^2 + 1/n^3} = 1 > 0$,

so $\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

(Note that $a_n > 0$ for $n \geq 6$.)

28. $\frac{e^{1/n}}{n} > \frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 > 0. \text{ Since } \sum_{n=1}^{\infty} b_n \text{ is the divergent harmonic series,}$$

$\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. [Note that we could also use l'Hospital's Rule to evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\cos(1/x) \cdot (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 = 1.]$$

34. $\sum_{n=1}^{10} \frac{\sin^2 n}{n^3} = \frac{\sin^2 1}{1} + \frac{\sin^2 2}{8} + \frac{\sin^2 3}{27} + \cdots + \frac{\sin^2 10}{1000} \approx 0.83253$. Now $\frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$, so the error is

$$R_{10} \leq T_{10} \leq \int_{10}^{\infty} \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2x^2} \right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2t^2} + \frac{1}{200} \right) = \frac{1}{200} = 0.005.$$

35. $\sum_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1+2^n} < \frac{1}{2^n}$, so the error is

$$R_{10} \leq T_{10} = \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2} \text{ [geometric series]} \approx 0.00098.$$

37. Since $\frac{d_n}{10^n} \leq \frac{9}{10^n}$ for each n , and since $\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1d_2d_3 \dots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim_{n \rightarrow \infty} \frac{1}{n^p \ln n} = \infty$. If $0 \leq p \leq 1$, then $n^p \ln n \leq n \ln n \Rightarrow$

$\frac{1}{n^p \ln n} \geq \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 11.3.21), so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison

Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum_{n=2}^{\infty} b_n$ converges, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges.

(Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.