4. The function $f(x) = 1/x^5$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{x^5}\,dx = \lim_{t\to\infty}\int_1^t x^{-5}\,dx = \lim_{t\to\infty}\left[\frac{x^{-4}}{-4}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{4t^4}+\frac{1}{4}\right) = \frac{1}{4}.$$

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty}\frac{1}{n^5}$ is also convergent by the Integral Test.

5. The function $f(x) = \dfrac{1}{(2x+1)^3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4}\,\frac{1}{(2x+1)^2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{4(2t+1)^2}+\frac{1}{36}\right) = \frac{1}{36}.$$

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}$ is also convergent by the Integral Test.

8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t\left(1+\frac{1}{x+1}\right)dx = \lim_{t\to\infty}\left[x+\ln(x+1)\right]_1^t = \lim_{t\to\infty}(t+\ln(t+1)-1-\ln 2) = \infty, \text{ so}$$

$\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\sum_{n=1}^{\infty}\frac{n+2}{n+1}$ is divergent.

Note: $\lim_{n\to\infty}\frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\sum_{n=1}^{\infty}\frac{1}{n^{0.85}}$ is a p-series with $p = 0.85 \le 1$, so it diverges by (1). Therefore, the series $\sum_{n=1}^{\infty}\frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\sum_{n=1}^{\infty}\frac{1}{n^{0.85}}$ would have to converge [by Theorem 8(i) in Section 11.2].

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

14. $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots = \sum_{n=1}^{\infty}\frac{1}{3n+2}$. The function $f(x) = \dfrac{1}{3x+2}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{3x+2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{3x+2}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{3}\ln|3x+2|\right]_1^t = \tfrac{1}{3}\lim_{t\to\infty}(\ln(3t+2)-\ln 5) = \infty, \text{ so the series}$$

$\sum_{n=1}^{\infty}\frac{1}{3n+2}$ diverges.

16. $f(x) = \dfrac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x \geq 2$, so we can use the Integral Test [note that f is *not* decreasing on $[1, \infty)$].

$\displaystyle\int_2^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{3}\ln(x^3+1)\right]_2^t = \tfrac{1}{3}\lim_{t\to\infty}\left[\ln(t^3+1) - \ln 9\right] = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^3+1}$.

20. The function $f(x) = \dfrac{1}{x^2-4x+5} = \dfrac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-2)^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}(x-2)\right]_2^t = \lim_{t\to\infty}\left[\tan^{-1}(t-2) - \tan^{-1} 0\right]$$
$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2-4n+5}$ converges. Of course, this means that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2-4n+5}$ converges too.

21. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln n}$ diverges.

24. $f(x) = \dfrac{x^2}{e^x} \quad\Rightarrow\quad f'(x) = \dfrac{e^x(2x) - x^2e^x}{(e^x)^2} = \dfrac{xe^x(2-x)}{(e^x)^2} = \dfrac{x(2-x)}{e^x} < 0$ for $x > 2$, so f is continuous, positive, and decreasing on $[3, \infty)$ and so the Integral Test applies.

$$\int_3^\infty f(x)\,dx = \lim_{t\to\infty}\int_3^t \frac{x^2}{e^x}\,dx \overset{(\star)}{=} \lim_{t\to\infty}\left[-e^{-x}(x^2+2x+2)\right]_3^t = -\lim_{t\to\infty}\left[e^{-t}(t^2+2t+2) - e^{-3}(17)\right] \overset{(\star\star)}{=} \frac{17}{e^3},$$

so the series $\displaystyle\sum_{n=3}^\infty \frac{n^2}{e^n}$ converges.

$(\star)$: $\displaystyle\int x^2e^{-x}\,dx \overset{97}{=} -x^2e^{-x} + 2\int xe^{-x}\,dx \overset{97}{=} -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\,dx\right)$
$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C = -e^{-x}(x^2+2x+2) + C.$$

$(\star\star)$: $\displaystyle\lim_{t\to\infty}\frac{t^2+2t+2}{e^t} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{2t+2}{e^t} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{2}{e^t} = 0.$

25. The function $f(x) = \dfrac{1}{x^3 + x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\int_1^\infty \frac{1}{x^3 + x}\,dx = \lim_{t \to \infty} \int_1^t \left[\frac{1}{x} - \frac{x}{1 + x^2}\right] dx = \lim_{t \to \infty} \left[\ln x - \frac{1}{2}\ln(1 + x^2)\right]_1^t = \lim_{t \to \infty} \left[\ln \frac{x}{\sqrt{1 + x^2}}\right]_1^t$$

$$= \lim_{t \to \infty} \left(\ln \frac{t}{\sqrt{1 + t^2}} - \ln \frac{1}{\sqrt{2}}\right) = \lim_{t \to \infty} \left(\ln \frac{1}{\sqrt{1 + 1/t^2}} + \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2$$

so the series $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + n}$ converges.

27. We have already shown (in Exercise 21) that when $p = 1$ the series $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$.

$f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2, \infty)$, and $f'(x) = -\dfrac{p + \ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t \to \infty} \left[\frac{(\ln x)^{1-p}}{1 - p}\right]_2^t \quad [\text{for } p \neq 1] = \lim_{t \to \infty} \left[\frac{(\ln t)^{1-p}}{1 - p} - \frac{(\ln 2)^{1-p}}{1 - p}\right]$$

This limit exists whenever $1 - p < 0 \quad \Leftrightarrow \quad p > 1$, so the series converges for $p > 1$.

31. Since this is a p-series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of real numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of real numbers x such that the series is convergent.