

1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

9. (a) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (11.1.1).

(b) Since $\lim_{n \rightarrow \infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

14. $1 + 0.4 + 0.16 + 0.064 + \dots$ is a geometric series with ratio $r = 0.4 = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges to

$$\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}.$$

16. $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}} = \sum_{n=1}^{\infty} \frac{10(10)^{n-1}}{(-9)^{n-1}} = 10 \sum_{n=1}^{\infty} \left(-\frac{10}{9}\right)^{n-1}$. The latter series is geometric with $a = 10$ and ratio $r = -\frac{10}{9}$.

Since $|r| = \frac{10}{9} > 1$, the series diverges.

23. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

28. $\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n$ [difference of two convergent geometric series]

$$= \frac{1}{1-0.8} - \frac{0.3}{1-0.3} = 5 - \frac{3}{7} = \frac{32}{7}$$

29. $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ diverges by the Test for Divergence since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n \rightarrow \infty} \frac{n^2+1}{2n^2+1}\right) = \ln \frac{1}{2} \neq 0.$$

30. $\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is

$$\frac{\cos 1}{1 - \cos 1} \approx 1.175343.$$

35. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ are

$$\begin{aligned} s_n &= \sum_{i=2}^n \frac{2}{(i-1)(i+1)} = \sum_{i=2}^n \left(\frac{1}{i-1} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) \end{aligned}$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

$$\text{Thus, } \sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}.$$

$$42. 0.\overline{73} = \frac{73}{10^2} + \frac{73}{10^4} + \cdots = \frac{73/10^2}{1 - 1/10^2} = \frac{73/100}{99/100} = \frac{73}{99}$$

$$47. \sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n \text{ is a geometric series with } r = \frac{x}{3}, \text{ so the series converges } \Leftrightarrow |r| < 1 \Leftrightarrow \frac{|x|}{3} < 1 \Leftrightarrow |x| < 3;$$

$$\text{that is, } -3 < x < 3. \text{ In that case, the sum of the series is } \frac{a}{1-r} = \frac{x/3}{1-x/3} = \frac{x/3}{1-x/3} \cdot \frac{3}{3} = \frac{x}{3-x}.$$

$$72. \text{ (a) RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1} + f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{1}{f_{n-1} f_{n+1}} = \text{LHS}$$

$$\begin{aligned} \text{(b) } \sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} &= \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \quad [\text{from part (a)}] \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5} \right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}} \right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1 \cdot 1} = 1 \quad \text{because } f_n \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

$$\begin{aligned} \text{(c) } \sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} &= \sum_{n=2}^{\infty} \left(\frac{f_n}{f_{n-1} f_n} - \frac{f_n}{f_n f_{n+1}} \right) \quad [\text{as above}] \\ &= \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right) \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{1}{f_1} - \frac{1}{f_3} \right) + \left(\frac{1}{f_2} - \frac{1}{f_4} \right) + \left(\frac{1}{f_3} - \frac{1}{f_5} \right) + \left(\frac{1}{f_4} - \frac{1}{f_6} \right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}} \right) = 1 + 1 - 0 - 0 = 2 \quad \text{because } f_n \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

73. (a) At the first step, only the interval $(\frac{1}{3}, \frac{2}{3})$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, which have a total length of $2 \cdot (\frac{1}{3})^2$. At the third step, we remove 2^2 intervals, each of length $(\frac{1}{3})^3$. In general, at the n th step we remove 2^{n-1} intervals, each of length $(\frac{1}{3})^n$, for a length of $2^{n-1} \cdot (\frac{1}{3})^n = \frac{1}{3} (\frac{2}{3})^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty} \frac{1}{3} (\frac{2}{3})^{n-1} = \frac{1/3}{1-2/3} = 1$ [geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$]. Notice that at the n th step, the leftmost interval that is removed is $(\frac{1}{3})^n, (\frac{2}{3})^n$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $(1 - (\frac{2}{3})^n, 1 - (\frac{1}{3})^n)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9},$ and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot (\frac{1}{9})^2$; at the third step, $(8)^2 \cdot (\frac{1}{9})^3$. In general, the area removed at the n th step is $(8)^{n-1} (\frac{1}{9})^n = \frac{1}{9} (\frac{8}{9})^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1-8/9} = 1.$$