

4. Since $f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$, Equation 6 gives the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!} (x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{3^n(n+1)n!} (x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)} (x-4)^n, \text{ which is the Taylor series for } f \text{ centered}$$

at 4. Apply the Ratio Test to find the radius of convergence R .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(-1)^n(x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)(x-4)(n+1)}{3(n+2)} \right| \\ &= \frac{1}{3} |x-4| \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = \frac{1}{3} |x-4| \end{aligned}$$

For convergence, $\frac{1}{3} |x-4| < 1 \Leftrightarrow |x-4| < 3$, so $R = 3$.

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
\vdots	\vdots	\vdots

$$\begin{aligned} (1-x)^{-2} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots \\ &= 1 + 2x + \frac{6}{2}x^2 + \frac{24}{6}x^3 + \frac{120}{24}x^4 + \dots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=0}^{\infty} (n+1)x^n \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+2}{n+1} = |x|(1) = |x| < 1$$

for convergence, so $R = 1$.

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
\vdots	\vdots	\vdots

$$\begin{aligned} \ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &\quad + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \frac{24}{120}x^5 - \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{1+1/n} = |x| < 1 \text{ for convergence,}$$

so $R = 1$.

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos 3x$	1
1	$-3 \sin 3x$	0
2	$-3^2 \cos 3x$	-3^2
3	$3^3 \sin 3x$	0
4	$3^4 \cos 3x$	3^4
\vdots	\vdots	\vdots

$$\cos 3x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}}{(2n)!} x^{2n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{2n+2} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{3^{2n} x^{2n}} \right| = \lim_{n \rightarrow \infty} \frac{3^2 x^2}{(2n+2)(2n+1)}$$

$$= 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x e^x$	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
\vdots	\vdots	\vdots

$$x e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[\frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n} \right] = \lim_{n \rightarrow \infty} \frac{|x|}{n} = 0 < 1 \quad \text{for all } x,$$

so $R = \infty$.

13.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0
6	0	0
\vdots	\vdots	\vdots

$f^{(n)}(x) = 0$ for $n \geq 5$, so f has a finite series expansion about $a = 1$.

$$f(x) = x^4 - 3x^2 + 1 = \sum_{n=0}^4 \frac{f^{(n)}(1)}{n!} (x-1)^n$$

$$= \frac{-1}{0!} (x-1)^0 + \frac{-2}{1!} (x-1)^1 + \frac{6}{2!} (x-1)^2 + \frac{24}{3!} (x-1)^3 + \frac{24}{4!} (x-1)^4$$

$$= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

A finite series converges for all x , so $R = \infty$.

27. $\frac{1}{(2+x)^3} = \frac{1}{[2(1+x/2)]^3} = \frac{1}{8} \left(1 + \frac{x}{2}\right)^{-3} = \frac{1}{8} \sum_{n=0}^{\infty} \binom{-3}{n} \left(\frac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdots(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdots[-(n+2)]}{n!}$$

$$= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n+1)(n+2)}{2 \cdot n!} = \frac{(-1)^n (n+1)(n+2)}{2}$$

Thus, $\frac{1}{(2+x)^3} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$, so $R = 2$.

$$30. \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(\pi x/2) = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi x/2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2^{2n} (2n)!} x^{2n}, R = \infty.$$

$$32. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow 2e^{-x} = 2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}, \text{ so } f(x) = e^x + 2e^{-x} = \sum_{n=0}^{\infty} \frac{[1 + 2(-1)^n]}{n!} x^n, R = \infty.$$

35. We must write the binomial in the form $(1 + \text{expression})$, so we'll factor out a 4.

$$\begin{aligned} \frac{x}{\sqrt{4+x^2}} &= \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2} \left(1 + \frac{x^2}{4}\right)^{-1/2} = \frac{x}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x^2}{4}\right)^n \\ &= \frac{x}{2} \left[1 + \left(-\frac{1}{2}\right) \frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} \left(\frac{x^2}{4}\right)^3 + \dots \right] \\ &= \frac{x}{2} + \frac{x}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot 4^n \cdot n!} x^{2n} \\ &= \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{3n+1}} x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \Leftrightarrow \frac{|x|}{2} < 1 \Leftrightarrow |x| < 2, \text{ so } R = 2. \end{aligned}$$

$$37. \sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} \left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right] = \frac{1}{2} \left[1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!},$$

$R = \infty$

$$47. \cos x \stackrel{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \Rightarrow$$

$$x \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \Rightarrow \int x \cos(x^3) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}, \text{ with } R = \infty.$$

$$49. \cos x \stackrel{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos x - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \frac{\cos x - 1}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!} \Rightarrow$$

$$\int \frac{\cos x - 1}{x} dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n \cdot (2n)!}, \text{ with } R = \infty.$$

$$64. \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n} (2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \text{ by (16).}$$