Optimal transportation between unequal dimensions

In this series of lectures, we introduce the Monge-Kantorovich problem of optimally transporting one distribution of mass onto another, where optimality is measured against a cost function $c(x, y) = -s(x, y)$. Connections to geometry, inequalities, and partial differential equations will be discussed, along with applications to economics. We survey recent developments, concentrating in particular on questions of uniqueness and regularity of solutions in a geometric setting.

For smooth costs and densities on compact manifolds, the only known examples for which the optimal solution is always unique require at least one of the two underlying spaces to be homeomorphic to a sphere. We shall describe a (multivalued) dynamics which the transportation cost induces between the target and source space, for which the presence or absence of a sufficiently large set of periodic trajectories plays a role in determining whether or not optimal transport is necessarily unique. This insight allowed Rifford and I to construct smooth costs on a pair of compact manifolds with arbitrary topology, so that the optimal transportation between any pair of probility densities is unique. Following work with Chiappori and Pass, we give conditions under which the solution to this problem can be reduced to the solution of a partial differential equation. However, in contrast to the case where the source and target have equal dimensions, this equation presents new challenges, being generally nonlocal.

See publications [46][52][60][61] at www.math.toronto.edu/mccann/publications