ALGEBRAIC DE RHAM COHOMOLOGY

Let $n > 6$ and let

$$f = x^n + y^n + sx^{n-2}y + txy^{n-2}$$

We're going to work in

$$A = \mathbf{Q}[x, y, s, t, \frac{1}{st-1}] = \mathbf{Q}[s, t, \frac{1}{st-1}][x, y]$$

Let $b \in B = \mathbf{Q}[s, t, \frac{1}{st-1}]$. Suppose we want to solve

$$(*) \qquad bf = g_1\frac{\partial f}{\partial x} + g_2\frac{\partial f}{\partial y} + g_3\frac{\partial f}{\partial s} + g_4\frac{\partial f}{\partial t} + \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial s} + \frac{\partial g_4}{\partial t}\right) f$$

for $g_i \in A$.

First observation: we get

$$(n-2)sg_1x^{n-3}y + (n-2)tg_2xy^{n-3} \equiv 0 \bmod (x^{n-1}, y^{n-1}, x^{n-2}y, xy^{n-2})$$

It follows that $g_1 \in (x, y^2)$ and $g_2 \in (y, x^2)$.

Say $h_1 \in B$ is the coefficient of x in g_1 and $h_2 \in B$ is the coefficient of y in g_2. Since s and t are invertible we can write the constant term of g_3 as sh_3 and the constant term of g_4 as th_4. Then we see that we obtain

$$bf \equiv nh_1x^n + nh_2y^n + ((n-2)h_1 + h_3)sx^{n-2}y + ((n-2)h_2 + h_4)txy^{n-2} +$$
$$(h_1 + h_2 + h_3 + s\frac{\partial h_3}{\partial s} + h_4 + t\frac{\partial h_4}{\partial t})f$$

modulo the ideal

$$(x^{n+1}, x^ny, x^{n-1}y^2, x^{n-2}y^3, y^{n+1}, xy^n, x^2y^{n-1}, x^3y^{n-2})$$

It follows that

$$nh_1 = nh_2 = (n-2)h_1 + h_3 = (n-2)h_2 + h_4$$

Setting $h = h_1$ we get $h_2 = h$, $h_3 = 2h$ and $h_4 = 2h$ and then we get the equation

$$b = (n+6)h + 2s\frac{\partial h}{\partial s} + 2t\frac{\partial h}{\partial t}$$

Now write

$$h = \frac{p(s,t)}{(st-1)^m}$$

with $m \geq 0$ minimal and $p(s,t)$ a polynomial. Then for $m > 0$ we get

$$(n+6)h + 2s\frac{\partial h}{\partial s} + 2t\frac{\partial h}{\partial t} = \frac{-4mstp}{(st-1)^{m+1}} + \frac{q}{(st-1)^m}$$

for some polynomial q.

Conclusion: if $b = (\sum a_i s^i)/(st-1)$ with $a_i \in \mathbf{Q}$, then we can solve (*) only if all a_i are 0.