1. Preliminaries

1.1. **Symmetric powers .** Let $\mathcal{C}$ be a $\mathbb{Q}$-linear Karoubian additive symmetric monoidal category. Let n be a nonnegative integer, and let S_n be the group of permutations of the set of integers from 1 to n. For any object M of $\mathcal{C}$, the group S_n acts on the left on $M^{\otimes n}$. We define $\mathrm{Sym}^n(M)$ as the image of the idempotent $p_n = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma$ of $M^{\otimes n}$. We have

$$\dim(\mathrm{Sym}^n(M)) = \mathrm{Tr}(p_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{Tr}(\sigma|M^{\otimes n}) = \frac{1}{n!}\prod_{0\leq j<n}(\dim(\mathrm{M})+j).$$

1.2. **Exterior powers .** Let $\mathcal{C}$ and n be as in 1.1. For any object M of $\mathcal{C}$, we define $\Lambda^n(M)$ as the image of the idempotent $a_n = \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\sigma$ of $M^{\otimes n}$, where $\varepsilon : S_n \to \{\pm 1\}$ is the signature homomorphism. We have

$$\dim(\Lambda^n(M)) = \mathrm{Tr}(a_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\mathrm{Tr}(\sigma|M^{\otimes n}) = \frac{1}{n!}\prod_{0\leq j<n}(\dim(\mathrm{M})-j).$$

2. Embedding result

2.1. **Deligne's condition .** An additive symmetric monoidal category $\mathcal{C}$, in which any object is dualizable, is said to satisfy **Deligne's condition** if the following assertions hold:

(1) for any object M of $\mathcal{C}$ with $\dim(M) = 0$ then $M = 0$;
(2) for any object M of $\mathcal{C}$ and any positive integer j, the endomorphism $\dim(M)+j$ is invertible;
(3) for any object M of $\mathcal{C}$, there exists an integer n such that $\prod_{0\leq j<n}(\dim(\mathrm{M})-j) = 0$.

2.2. **Rigid exact symmetric monoidal categories .** A **rigid exact** symmetric monoidal category is an additive symmetric monoidal category $\mathcal{C}$, in which any object is dualizable, together with a class Seq of sequences $M_1 \to M_2 \to M_3$ of homomorphisms of $\mathcal{C}$ such that:

(1) the couple $(\mathcal{C}, \mathrm{Seq})$ is an exact category;
(2) for any object M of $\mathcal{C}$, the functor $M \otimes -$ sends Seq to itself;
(3) the duality functor from $\mathcal{C}$ to itself sends Seq to itself.

In particular, the dual of an admissible monomorphism is an admissible epimorphism, and conversely.

Example 2.3. *For any ring R, the category Vect_R of finite projective R-modules is naturally a rigid exact symmetric monoidal category. Moreover, it satisfies Deligne's condition 2.1 if and only if R is a $\mathbb{Q}$-algebra.*

Theorem 2.4. *Let $\mathcal{C}$ be a rigid exact symmetric monoidal category which is $\mathbb{Q}$-linear, Karoubian and essentially small. Then the following are equivalent:*

(i) *Deligne's condition 2.1 holds for $\mathcal{C}$;*
(ii) *there exists a $\mathbb{Q}$-algebra R and a faithful exact monoidal functor from $\mathcal{C}$ to Vect_R.*

3. Splitting lemmas

Let $\mathcal{C}$ be as in Theorem 2.4.

3.1. **Faithfulness.** A commutative algebra A in $\mathrm{Ind}(\mathcal{C})$ is said to be **faithful** (or **faithfully flat**) if the unit morphism $1 \to A$ is an admissible monomorphism.

Example 3.2. *If $\mathcal{C} = \mathrm{Vect}_R$ for some ring R then $\mathrm{Ind}(\mathcal{C})$ is the symmetric monoidal category of flat R-modules by Lazard's theorem [058G]. Thus a commutative algebra in $\mathrm{Ind}(\mathcal{C})$ is simply a flat homomorphism $R \to A$ of rings. It is faithful as an object of $\mathrm{Ind}(\mathcal{C})$ if and only if $R \to A$ is injective with flat cokernel, or equivalently if A is a faithfully flat R-algebra.*

Proposition 3.3. *Let M be an object of $\mathcal{C}$ and let $M \xrightarrow{\lambda} 1$ be an admissible epimorphism. Let us consider the commutative algebra*

$$A = \mathrm{colim}_n\ \mathrm{Sym}^n(M^\vee),$$

in $\mathrm{Ind}(\mathcal{C})$, with transition maps given by the multiplication by $1 \xrightarrow{\lambda^\vee} M^\vee$. Then we have the following assertions:

1. *the epimorphism $A \otimes M \xrightarrow{\mathrm{id}_A \otimes v} A$ of A-modules admits an A-linear section;*
2. *the unit homomorphism $1 \to A$ is an admissible monomorphism, i.e. A is faithful.*

Proof. (1) The canonical morphism $M^\vee \to A$ yields an A-linear morphism $A \to A \otimes M$, and its composition with λ is the identity of A by construction.

(2) The composition of the morphism $\mathrm{Sym}^n(M) \to 1$ induced by v with the projection $M^{\otimes n} \xrightarrow{p_n} \mathrm{Sym}^n(M)$ is the admissible epimorphism $\lambda^{\otimes n}$. Thus $\mathrm{Sym}^n(M) \to 1$ is an admissible epimorphism, and its dual $1 \to \mathrm{Sym}^n(M^\vee)$ is consequently an admissible monomorphism in $\mathcal{C}$. Hence the colimit $1 \to A$ is an admissible monomorphism in $\mathrm{Ind}(\mathcal{C})$.

□

Corollary 3.4. *Let $M \to N$ be an admissible epimorphism in $\mathcal{C}$. Then there exists a faithful commutative algebra A in $\mathrm{Ind}(\mathcal{C})$ such that the admissible epimorphism $A \otimes M \to A \otimes N$ splits.*

Proof. For any commutative algebra A in $\mathrm{Ind}(\mathcal{C})$, the epimorphism $A \otimes M \to A \otimes N$ splits if and only if the corresponding morphism $A \otimes M \otimes N^\vee \to A$ splits. The morphism $M \otimes N^\vee \to 1$ is the pullback along the coevaluation $1 \to N \otimes N^\vee$ of the admissible epimorphism $M \otimes N^\vee \to N \otimes N^\vee$, hence is itself admissible. The conclusion then follows from Lemma 3.3 applied to $M \otimes N^\vee \to 1$.

□

Proposition 3.5. *Let M be an object of $\mathcal{C}$ and let us consider the commutative algebra*

$$A = \mathrm{colim}_n\ \mathrm{Sym}^n(M^\vee \oplus M),$$

in $\mathrm{Ind}(\mathcal{C})$, with transition maps $\mathrm{Sym}^n(M^\vee \oplus M) \to \mathrm{Sym}^{n+2}(M^\vee \oplus M)$ given by the multiplication by the coevaluation $1 \xrightarrow{\mathrm{coev}} M^\vee \otimes M$. Then we have the following assertions:

1. *the A-module $A \otimes M$ admits a direct summand which is a free A-module of rank 1;*
2. *if $\dim(M) + j$ is invertible for any nonnegative integer j, then the unit homomorphism $1 \to A$ is a split monomorphism. In particular, A is faithful.*

Proof. (1) The canonical morphisms $M^\vee \xrightarrow{u_0} A$ and $M \xrightarrow{v_0} A$ induce A-module homomorphisms $A \xrightarrow{u} A \otimes M$ and $A \otimes M \xrightarrow{v} A$. The composition $v \circ u$ coincides with the A-linearization of the composition

$$1 \xrightarrow{\mathrm{coev}} M^\vee \otimes M \xrightarrow{u_0 \otimes v_0} A,$$

which is simply the unit $1 \to A$ by construction of A. Thus $v \circ u$ is the identity of A, and consequently u is a split monomorphism of A-modules.

(2) The algebra A admits the subalgebra

$$B = \operatorname{colim}_n \operatorname{Sym}^n(M^\vee) \otimes \operatorname{Sym}^n(M),$$

as a direct summand, where the transitions are again given by multiplying by the coevaluation $1 \xrightarrow{\text{coev}} M^\vee \otimes M$. Let e_n be the composition

$$\operatorname{Sym}^n(M^\vee) \otimes \operatorname{Sym}^n(M) \to (M^\vee)^{\otimes n} \otimes M^{\otimes n} \xrightarrow{\text{ev}^{\otimes n}} 1.$$

Let d_1 be the dimension of M and let $d_n = \frac{1}{n!} d_1(d_1+1)\cdots(d_1+n-1)$ be the dimension of $\operatorname{Sym}^n(M)$. By assumption, each d_n is an invertible endomorphism of the unit 1. The composition

$$\operatorname{Sym}^n(M^\vee) \otimes \operatorname{Sym}^n(M) \xrightarrow{\text{coev}} \operatorname{Sym}^{n+1}(M^\vee) \otimes \operatorname{Sym}^{n+1}(M) \xrightarrow{d_{n+1}^{-1} e_{n+1}} 1,$$

coincides with $d_n^{-1} e_n$, hence the morphisms $(d_n^{-1} e_n)_n$ provide a morphism $B \to 1$, which splits the unit of B.

□

Corollary 3.6. *Let M be an object of $\mathcal{C}$ and let d be a nonnegative integer such that* $\dim(M) + j$ *is invertible for any integer $j > -d$. Then there exists a faithful commutative algebra A in* $\operatorname{Ind}(\mathcal{C})$ *such that the A-module $A \otimes M$ admits a direct summand which is a free A-module of rank d.*

Proof. For $d = 0$, we simply take $A = 1$, the unit object of $\mathcal{C}$. We prove the general case by induction. Let us assume that d is positive and that the assertion is proved for $d-1$. By Lemma 3.5, we have a faithful commutative algebra B in $\mathcal{C}$ such that the B-module $B \otimes M$ is isomorphic to $B \oplus N$ for some B-module N. Since N is a direct summand of the dualizable B-module $B \otimes N$, it is a dualizable B-module as well. The rigid exact symmetric monoidal category of dualizable B-modules satisifies the assumptions of Corollary 3.6, and its object N is dualizable with dimension $\dim(N) = \dim(M) - 1$ such that $\dim(N) + j$ is invertible for any integer $j > -d+1$. By induction there exists a faithful commutative B-algebra A in $\operatorname{Ind}(\mathcal{C})$ such that the A-module $A \otimes_B N$ admits a free direct summand of rank $d-1$. Then A satisfies the conclusion of Corollary 3.6 for M. □

4. Proof of Theorem 2.4

Clearly (ii) implies (i) in Theorem 2.4. Conversely, let $\mathcal{C}$ be as in Theorem 2.4, and let us assume that Deligne's condition holds for $\mathcal{C}$. Let M be an object of $\mathcal{C}$. By assumption, there exists an integer n and orthogonal idempotents $(e_d)_{d=0}^n$ of $\operatorname{End}(1)$ such that $\dim(M) e_d = d e_d$ and $\sum_d e_d = 1$. By Proposition 3.6, there exists a faithful commutative algebra A in $\operatorname{Ind}(\mathcal{C})$ such that for any integer d, the $e_d A$-module $e_d A \otimes e_d M$ is of the form $(e_d A)^{\oplus d} \oplus N_d$. The $e_d A$-module

$$\Lambda^{d+1}_{e_d A}(e_d A \otimes e_d M)) = e_d A \otimes \Lambda^{d+1}(e_d M),$$

contains $\Lambda^d_{e_d A}(e_d A^{\oplus d}) \otimes N \simeq N$ as a direct summand, and vanishes since $\Lambda^{d+1}(e_d M)$ has vanishing dimension. Thus N vanishes, and $e_d A \otimes e_d M$ is isomorphic to $(e_d A)^{\oplus d}$.

Since $\mathcal{C}$ is essentially small, and since the category of faithful algebras in $\operatorname{Ind}(\mathcal{C})$ is cocomplete, we can assume (and we do) that the same algebra A has these properties for all objects M of $\mathcal{C}$. Likewise by Proposition 3.3 there exists a faithful commutative A-algebra B in $\operatorname{Ind}(\mathcal{C})$ such that $A \to B$ is an admissible monomorphism and such that any short exact sequence in $\mathcal{C}$ splits over B.

Let R be the ring $\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(1, B)$ and let us consider the additive functor F which sends an object M of $\mathcal{C}$ to the R-module $\operatorname{Hom}_{\operatorname{Ind}(C)}(1, B \otimes M)$. Since any short exact sequence of $\mathcal{C}$ splits over B, the functor F is exact. Moreover, for any object M of $\mathcal{C}$, with $(e_d)_d$ as above, we have

$$F(M) = F\left(\oplus_{d=0}^n e_d M\right) = \oplus_{d=0}^n F(e_d M) \simeq \oplus_{d=0}^n F(e_d) R^{\oplus d},$$

hence $F(M)$ is a finite projective R-module. A similar computation shows that the natural transformation $F(M_1) \otimes_R F(M_2) \to F(M_1 \otimes M_2)$ is an isomorphism. Since we also have $F(1) = R$, the functor F is monoidal.

It remains to prove that F is faithful. Let M_1, M_2 be objects of $\mathcal{C}$. The fact that $1 \to B$ is an admissible monomorphism, together with the left exactness of the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(1, M_1^\vee \otimes M_2 \otimes -)$ on $\mathrm{Ind}(\mathcal{C})$, yields that

$$\mathrm{Hom}_{\mathcal{C}}(M_1, M_2) = \mathrm{Hom}_{\mathcal{C}}(1, M_1^\vee \otimes M_2) \subseteq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(1, M_1^\vee \otimes M_2 \otimes B),$$

and the last term, namely $F(M_1^\vee \otimes M_2)$, is canonically isomorphic to $\mathrm{Hom}_{\mathrm{Vect}_R}(F(M_1), F(M_2))$ since F is monoidal.