1. (Multi)linear algebra in symmetric monoidal categories

1.1. **Dimension.**

1.2. **Symmetric powers .** Let $\mathcal{C}$ be a $\mathbb{Q}$-linear Karoubian additive symmetric monoidal category. Let n be a nonegative integer, and let S_n be the group of permutations of the set of integers from 1 to n. For any object M of $\mathcal{C}$, the group S_n acts on the left on $M^{\otimes n}$. We define $\mathrm{Sym}^n(M)$ as the image of the idempotent $p_n = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma$ of $M^{\otimes n}$. We have

$$\dim(\mathrm{Sym}^n(M)) = \mathrm{Tr}(p_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{Tr}(\sigma|M^{\otimes n}) = \frac{1}{n!}\prod_{0\leq j<n}(\dim(\mathrm{M})+j).$$

1.3. **Exterior powers .** Let $\mathcal{C}$ and n be as in 1.2. For any object M of $\mathcal{C}$, we define $\Lambda^n(M)$ as the image of the idempotent $a_n = \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\sigma$ of $M^{\otimes n}$, where $\varepsilon : S_n \to \{\pm 1\}$ is the signature homomorphism. We have

$$\dim(\Lambda^n(M)) = \mathrm{Tr}(a_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon(\sigma)\mathrm{Tr}(\sigma|M^{\otimes n}) = \frac{1}{n!}\prod_{0\leq j<n}(\dim(\mathrm{M})-j).$$

2. Embedding result

2.1. **Deligne's condition .** An additive symmetric monoidal category $\mathcal{C}$ is said to satisfy **Deligne's condition** if the following assertions hold:

(1) for any object M of $\mathcal{C}$ with $\dim(M) = 0$ then $M = 0$;
(2) for any object M of $\mathcal{C}$ and any positive integer j, the endomorphism $\dim(M)+j$ is invertible;
(3) for any object M of $\mathcal{C}$, there exists an integer n such that $\prod_{0\leq j<n}(\dim(\mathrm{M})-j) = 0$.

2.2. **Exact symmetric monoidal categories .** An exact symmetric monoidal category is an additive symmetric monoidal category $\mathcal{C}$ together with a class Seq of sequences $M_1 \to M_2 \to M_3$ of homomorphisms of $\mathcal{C}$ such that:

(1) the couple $(\mathcal{C}, \mathrm{Seq})$ is an exact category;
(2) for any object M of $\mathcal{C}$, the functor $M \otimes -$ sends Seq to itself.

Example 2.3. *For any ring R, the category Vect_R of finite projective R-modules is naturally an exact symmetric monoidal category. Moreover, it satisfies Deligne's condition 2.1 if and only if R is a $\mathbb{Q}$-algebra.*

Theorem 2.4. *(??) Let $\mathcal{C}$ be an exact symmetric monoidal category, such that:*

(1) *the category $\mathcal{C}$ is $\mathbb{Q}$-linear, Karoubian and essentially small;*
(2) *any object of $\mathcal{C}$ is dualizable.*

Then the following are equivalent:

- *Deligne's condition 2.1 holds for $\mathcal{C}$;*
- *there exists a $\mathbb{Q}$-algebra R and a faithful exact monoidal functor from $\mathcal{C}$ to Vect_R.*

3. A few thoughts

Let $\mathcal{C}$ be as in Theorem 2.4. Let us consider the groupoid-valued presheaf

$$\mathcal{G} : A \mapsto \{\mathcal{C} \xrightarrow{F} \mathrm{Vect}_A \text{ exact monoidal (unit-preserving) functor}\},$$

on the category of rings.

Example 3.1. *If* $\mathcal{C} = \mathrm{Vect}_R$ *for some ring* R*, then* $\mathcal{G}$ *is representable by* $\mathrm{Spec}(R)$*. Presumably the same happens if we replace* R *with a quasicompact quasisperated scheme.*

The question is, can we recover $\mathcal{C}$ from the groupoid $\mathcal{G}$? Also, under the hypotheses of Theorem 2.4, what can be said on $\mathcal{G}$? Is it an algebraic stack? Perhaps one should assume $\mathcal{C}$ to be finitely generated.