REMARKS :

1) In Proposition 0.1(2) below, is $\mathbb{Q}$-linearity necessary? More generally, is there a relation between the properties "being colimit of dualizable" and "being flat" in general?

2) We should include examples of $\mathbb{Q}$-linear symmetric monoidal categories with non integral dimensions (Deligne categories?). Same for nonzero objects with zero dimension.

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Let $\mathcal{C}$ be a symmetric monoidal abelian category which has all small colimits, i.e. which is cocomplete. For any object M of $\mathcal{C}$, the graded commutative ring

$$\mathrm{Sym}^\bullet(M) = \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n(M),$$

is defined as largest commutative quotient of the tensor algebra $T(M) = \bigoplus_{n\in\mathbb{N}} M^{\otimes n}$.

Proposition 0.1. *Let $\mathcal{C}$ be a cocomplete symmetric monoidal abelian category. Let M be a dualizable object of $\mathcal{C}$ and let us consider the commutative algebra*

$$A = \mathrm{colim}_n\ \mathrm{Sym}^n(M^\vee \oplus M),$$

in $\mathcal{C}$, with transition maps $\mathrm{Sym}^n(M^\vee \oplus M) \to \mathrm{Sym}^{n+2}(M^\vee \oplus M)$ given by the multiplication by the coevaluation $1 \xrightarrow{\mathrm{coev}} M^\vee \otimes M$. Then we have the following assertions:

(1) *the A-module $A \otimes M$ admits a direct summand which is a free A-module of rank 1;*
(2) *if $\mathcal{C}$ is $\mathbb{Q}$-linear then A is a flat algebra;*
(3) *if $\mathcal{C}$ is $\mathbb{Q}$-linear and if $\dim(M) + j$ is invertible for any nonnegative integer j, then the unit homomorphism $1 \to A$ is a split monomorphism, and the algebra A is consequently faithfully flat.*

Proof. (1) The canonical morphisms $M^\vee \xrightarrow{u_0} A$ and $M \xrightarrow{v_0} A$ induce A-module homomorphisms $A \xrightarrow{u} A \otimes M$ and $A \otimes M \xrightarrow{v} A$. The composition $v \circ u$ coincides with the A-linearization of the composition

$$1 \xrightarrow{\mathrm{coev}} M^\vee \otimes M \xrightarrow{u_0 \otimes v_0} A,$$

which is simply the unit $1 \to A$ by construction of A. Thus $v \circ u$ is the identity of A, and consequently u is a split monomorphism of A-modules.

(2) Let $N = M^\vee \oplus M$. For each nonnegative integer n, the group S_n of permutations of the first n positive integers acts on the left on the tensor power $N^{\otimes n}$ by permuting the factors. The endomorphism $p_n = \frac{1}{n!}\sum_{\sigma\in S_n} \sigma$ of $N^{\otimes n}$ is a projection, whose image maps isomorphically onto $\mathrm{Sym}^n(N)$. Thus $\mathrm{Sym}^n(N)$ is isomorphic to a direct summand of the dualizable object $N^{\otimes n}$, hence is dualizable and in particular flat. The algebra A is then a colimit a flat objects, hence is flat as well.

(3) The algebra A admits the subalgebra

$$B = \mathrm{colim}_n\ \mathrm{Sym}^n(M^\vee) \otimes \mathrm{Sym}^n(M),$$

as a direct summand, where the transitions are again given by multiplying by the coevaluation $1 \xrightarrow{\mathrm{coev}} M^\vee \otimes M$. Let $\iota_n : (M^{\otimes n})^{S_n} \to \mathrm{Sym}^n(M)$ and $\iota_n^\vee : (M^{\vee\otimes n})^{S_n} \to \mathrm{Sym}^n(M^\vee)$ be the restrictions of the natural surjections. Let e_n be the composition

$$\mathrm{Sym}^n(M^\vee) \otimes \mathrm{Sym}^n(M) \xrightarrow{(\iota_n^\vee)^{-1}\otimes\iota_n^{-1}} (M^\vee)^{\otimes n} \otimes M^{\otimes n} \xrightarrow{\mathrm{ev}^{\otimes n}} 1.$$

Let d_1 be the dimension of M and let $d_n = \frac{1}{n!}d_1(d_1+1)\cdots(d_1+n-1)$ be the dimension of $\mathrm{Sym}^n(M)$. By assumption, each d_n is an invertible endomorphism of the unit 1. The composition

$$\mathrm{Sym}^n(M^\vee) \otimes \mathrm{Sym}^n(M) \xrightarrow{\mathrm{coev}} \mathrm{Sym}^{n+1}(M^\vee) \otimes \mathrm{Sym}^{n+1}(M) \xrightarrow{d_{n+1}^{-1} e_{n+1}} 1,$$

coincides with $d_n^{-1} e_n$ [COMPUTATION TO BE EXPANDED], hence the morphisms $(d_n^{-1} e_n)_n$ provide a splitting $B \to 1$ of the unit of B. □

Corollary 0.2. *Let $\mathcal{C}$ be a cocomplete $\mathbb{Q}$-linear symmetric monoidal abelian category. Let M be a dualizable object of $\mathcal{C}$ and let d be a nonnegative integer such that $\dim(M) + j$ is invertible for any integer $j > -d$. Then there exists a faithfully flat commutative algebra A in $\mathcal{C}$ such that the A-module $A \otimes M$ admits a direct summand which is a free A-module of rank d.*

Proof. For $d = 0$, we simply take $A = 1$, the unit object of $\mathcal{C}$. We prove the general case by induction. Let us assume that d is positive and that the assertion is proved for $d - 1$. By Lemma 0.1, we have a faithfully flat algebra B in $\mathcal{C}$ such that the B-module $B \otimes M$ is isomorphic to $B \oplus N$ for some B-module N. Since N is a direct summand of the dualizable B-module $B \otimes N$, it is a dualizable B-module as well. The category of B-modules is a cocomplete $\mathbb{Q}$-linear symmetric monoidal abelian category, and its object N is dualizable with dimension $\dim(M) - 1$, and $\dim(M) - 1 + j$ is invertible for any integer $j > -d + 1$. By induction there exists a faithfully flat B-algebra A such that the A-module $A \otimes_B N$ admits a free direct summand of rank $d - 1$. Then A is a faithfully flat algebra in $\mathcal{C}$ and the A-module $A \otimes M$ admits a free direct summand of rank d. □

Corollary 0.3. *Let $\mathcal{C}$ be a cocomplete $\mathbb{Q}$-linear symmetric monoidal abelian category. Let $\mathcal{D}$ be an essentially small full subcategory of $\mathcal{C}$ whose objects are all dualizable in $\mathcal{C}$ with dimensions in $\mathbb{Z}_{\geq 0}$. Then there exists a faithfully flat commutative algebra A in $\mathcal{C}$ such that for any object M of $\mathcal{D}$, the A-module $A \otimes M$ admits a direct summand which is a free A-module of rank $\dim(M)$.*

Proof. Let D be a set of objects of $\mathcal{D}$ such that any object of $\mathcal{D}$ is isomorphic to an element of D. For each element M of D, by Corollary 0.2 we may take a faithfully flat algebra A_M in $\mathcal{C}$ such that $A_M \otimes M$ admits a direct summand which is a free A_M-module of rank d. The conclusion follows by taking A to be the coproduct of $(A_M)_{M \in D}$ in the category of commutative algebras of $\mathcal{C}$. □