NOTES ON TANNAKIAN DUALITY

Contents

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Convention. For a fixed (commutative) ring k, we use the term *tensor category* to refer to a k-linear symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$. A k-linear symmetric monoidal functor between such will be called a *tensor functor*. A mapping between tensor functors is a natural transformation that is compatible with the k-linear symmetric monoidal structures.

Summary. These notes concern two kinds of objects: affine category schemes $\mathcal{X}$ (= category objects in affine schemes) and triples $(\mathcal{C}, R, F)$, where:

(1) $\mathcal{C}$ is a (small) tensor category;
(2) R is a (commutative) k-algebra;
(3) F is a tensor functor from $\mathcal{C}$ to the tensor category $R\text{-Mod}^{\square}$ of dualizable R-modules.

We do not define a "Tannakian category." Whatever this notion is, it should be a triple $(\mathcal{C}, R, F)$ as above satisfying certain conditions.

We explain two constructions. Given an affine category scheme $\mathcal{X}$, we associate to it a triple where $\mathcal{C}$ is the category of "finite-rank representations" of $\mathcal{X}$, and F is the forgetful functor of "taking underlying vector space." Conversely, given a triple $(\mathcal{C}, R, F)$, we associate to it an affine category scheme $\mathcal{X}$.

Tannakian reconstruction poses the following question: give necessary and sufficient conditions on affine category schemes $\mathcal{X}$ and triples $(\mathcal{C}, R, F)$ for the above constructions to be inverses of each other. These notes do not contain a satisfactory answer to this question. Instead, we give:

(1) a sufficient condition on affine category schemes to be reconstructed. This includes transitive affine groupoids (as studied in [Deligne, Catégories Tannakiennes]), affine monoids over a field, and some non-transitive groupoids (whose representation category is not abelian).
(2) a sufficient condition on $(\mathcal{C}, R, F)$ for the *large category* $\mathrm{Ind}(\mathcal{C})$ to be reconstructed. This condition involves abelian-ness of $\mathcal{C}$ and seems rather restrictive.

We have not yet investigated in any detail the reconstruction of the small category $\mathcal{C}$ itself.

1. Affine category schemes

We fix a ground ring k. In this section, we define coalgebroids and affine category schemes, their geometric counterparts. Of particular importance is the category of finite-rank representations $\mathcal{X}\text{-Rep}^f$ of an affine category schemes $\mathcal{X}$.

The main result of this section is that an affine category scheme $\mathcal{X}$, whose regular representation is a filtered colimit of finite-rank ones, can be reconstructed from $\mathcal{X}\text{-Rep}^f$.

1.1. **Coalgebroids.**

1.1.1. A *coalgebroid* (over k) consists of a commutative k-algebra R^0, a commutative $(R^0 \otimes R^0)$-algebra R^1 where we view the two R^0-structures as multiplications on the left and right, together with the following additional data:

(1) an $(R^0 \otimes R^0)$-algebra map $\Delta : R^1 \to R^1 \underset{R^0}{\otimes} R^1$, called *comultiplication*;
(2) an $(R^0 \otimes R^0)$-algebra map $\varepsilon : R^1 \to R^0$, called *counit*,

such that the associative and unital conditions are satisfied.

1.1.2. A coalgebroid is the dual notion of an affine category scheme (i.e., a category object in affine schemes).

Indeed, writing $X^0 := \mathrm{Spec}(R^0)$ and $X^1 := \mathrm{Spec}(R^1)$, we obtain a diagram:

$$X^1 \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X^0.$$

Here, the projection pr_1 (resp. pr_2) corresponds to the left (resp. right) R^0-algebra structure on R^1. Furthermore, we obtain a composition law m and a unit e, satisfying associative and unital conditions. The composition law can be conveniently expressed by the following diagram:

$$\begin{tikzcd}
 & & X^1 \underset{X^0}{\times} X^1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "m"] & & X^1 \arrow[dd, "\mathrm{pr}_2"] \arrow[ddllll, "\mathrm{pr}_1"'] \\
 & X^1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[rr, phantom, "\square"] & & X^1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
X^0 & & X^0 & & X^0
\end{tikzcd}$$

where the middle square is Cartesian. The unit can be expressed by the following diagram:

$$\begin{tikzcd}
X^0 \arrow[r, "e"] \arrow[d, "\mathrm{id}"'] \arrow[drr, "\mathrm{id}"'] & X^1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
X^0 & & X^0
\end{tikzcd}$$

Remark 1.1. Informally, we view a point of X^0 as an object x and a point of X^1 as an arrow $f : x \to x'$ where $x = \mathrm{pr}_1(f)$ and $x' = \mathrm{pr}_2(f)$.

1.1.3. Given a coalgebroid $R^\bullet = (R^0, R^1, \Delta, \varepsilon)$, an $\mathcal{R}$-*comodule* consists of:

(1) an R^0-module $\mathcal{M}$;
(2) an R^0-module map $\rho : \mathcal{M} \to R^1 \underset{R^0}{\otimes} \mathcal{M}$ (for the second object, R^0 acts on R^1 on the left), called *coaction*,

such that the associative and unital conditions are satisfied; namely, the following two compositions are equal:

$$\mathcal{M} \xrightarrow{\rho} R^1 \underset{R^0}{\otimes} \mathcal{M} \underset{\mathrm{id}_{R^1} \otimes \rho}{\overset{\Delta \otimes \mathrm{id}_{\mathcal{M}}}{\rightrightarrows}} R^1 \underset{R^0}{\otimes} R^1 \underset{R^0}{\otimes} \mathcal{M},$$

and the following composition is $\mathrm{id}_{\mathcal{M}}$:

$$\mathcal{M} \xrightarrow{\rho} R^1 \underset{R^0}{\otimes} \mathcal{M} \xrightarrow{\varepsilon \otimes \mathrm{id}_{\mathcal{M}}} \mathcal{M}.$$

A morphism of $\mathcal{R}$-comodules is a map $\varphi : \mathcal{M} \to \mathcal{M}'$ of R^0-modules making the following diagram commute:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\varphi"] \arrow[d, "\rho_{\mathcal{M}}"] & \mathcal{M}' \arrow[d, "\rho_{\mathcal{M}'}"] \\ R^1 \underset{R^0}{\otimes} \mathcal{M} \arrow[r, "\mathrm{id}_{R^1} \otimes \varphi"] & R^1 \underset{R^0}{\otimes} \mathcal{M}' \end{tikzcd}$$

Thus $\mathcal{R}$-comodules form a category, denoted by $\mathcal{R}$-Comod.

In terms of the geometric description in §1.1.2, the datum ρ is equivalent to a morphism:

$$\rho : \mathcal{M} \to (\mathrm{pr}_1)_* \mathrm{pr}_2^* \mathcal{M},$$

which is compatible with m and e in the obvious sense. One can alternatively view ρ as a morphism $\mathrm{pr}_1^* \mathcal{M} \to \mathrm{pr}_2^* \mathcal{M}$ by adjunction.

Remark 1.2. Informally, the datum of an $\mathcal{R}$-comodule consists a module $\mathcal{M}_x$ for any point x of X^0 and a morphism $f_* : \mathcal{M}_x \to \mathcal{M}_{x'}$ for every point $f : x \to x'$ of X^1.

1.1.4. The comultiplication Δ equips the R^0-module R^1 (via left multiplication) the structure of an $\mathcal{R}$-comodule. We refer to it as the *regular representation*.

More generally, given any R^0-module $\mathcal{N}$, the R^0-module $R^1 \underset{R^0}{\otimes} \mathcal{N}$ together with comultiplication $\Delta \otimes \mathrm{id}_{\mathcal{N}}$ forms an $\mathcal{R}$-comodule. We call this the coinduced comodule of $\mathcal{N}$ and denote it by $\mathrm{Coind}(\mathcal{N})$.

Lemma 1.3. *The following pair of functors forms an adjunction:*

$$\mathrm{oblv} : \mathcal{R}\text{-Comod} \rightleftarrows R^0\text{-Mod} : \mathrm{Coind},$$

where oblv *is conservative and* Coind *commutes with all colimits.*

Proof. Given an $\mathcal{R}$-comodule $\mathcal{M}$ and an R^0-module $\mathcal{N}$, we construct a natural bijection:

$$\mathrm{Hom}_{R^0\text{-Mod}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{R}\text{-Comod}}(\mathcal{M}, R^1 \underset{R^0}{\otimes} \mathcal{N})$$

as follows. It associates to an R^0-module map $\varphi : \mathcal{M} \to \mathcal{N}$ the morphism:

$$\mathcal{M} \xrightarrow{\rho_{\mathcal{M}}} R^1 \underset{R^0}{\otimes} \mathcal{M} \xrightarrow{\mathrm{id}_{R^1} \otimes \varphi} R^1 \underset{R^0}{\otimes} \mathcal{M},$$

where $\rho_{\mathcal{M}}$ is the $\mathcal{R}$-comodule structure on $\mathcal{M}$. One checks that this is a map of $\mathcal{R}$-comodules using the compatibility of $\rho_{\mathcal{M}}$ with comultiplication Δ. The inverse of this construction associates to every $\mathcal{R}$-comodule map $\mathcal{M} \to R^1 \underset{R^0}{\otimes} \mathcal{N}$ its post-composition with $\varepsilon \otimes \mathrm{id}_{\mathcal{N}} : R^1 \underset{R^0}{\otimes} \mathcal{N} \to \mathcal{N}$.

The fact that oblv is conservative is obvious from the definition. Thus, in order to check that Coind preserves all colimits, it suffices to check that oblv $\circ$ Coind preserves all colimits. The latter follows from the fact that the tensor operation $R^1 \underset{R^0}{\otimes} - : R^0\text{-Mod} \to R^0\text{-Mod}$ commutes with all colimits. $\square$

1.1.5. We will now characterize the compact objects in $\mathcal{R}$-Comod.

Lemma 1.4. *Let $\mathcal{R} = (R^0, R^1, \Delta, \varepsilon)$ be a coalgebroid. The following are equivalent for any object $\mathcal{M} \in \mathcal{R}$-Comod:*

(1) $\mathcal{M}$ is compact;
(2) $\mathrm{oblv}(\mathcal{M}) \in R^0$-Mod is compact.

The second condition means concretely that $\mathrm{oblv}(\mathcal{M})$ is a finitely presented R^0-module.

(1) $\implies$ (2). By Lemma 1.3, oblv admits a right adjoint sending $\mathcal{N}$ to the $\mathcal{R}$-comodule $R^1 \underset{R^0}{\otimes} \mathcal{N}$. Since this functor commutes with filtered colimits, the left adjoint oblv preserves compact objects. $\square$

(2) $\implies$ (1). Suppose $\mathrm{oblv}(\mathcal{M}) \in \mathcal{R}^0$-Mod is compact. Let $\mathcal{M}'_i$ $(i \in \mathcal{I})$ be a filtered system in $\mathcal{R}$-Comod with colimit $\mathcal{M}'$. We have a commutative diagram:

$$\begin{tikzcd} \underset{i\in\mathcal{I}}{\mathrm{colim}}\, \mathrm{Hom}_{\mathcal{R}\text{-Comod}}(\mathcal{M}, \mathcal{M}'_i) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}_{\mathcal{R}\text{-Comod}}(\mathcal{M}, \mathcal{M}') \arrow[d] \\ \underset{i\in\mathcal{I}}{\mathrm{colim}}\, \mathrm{Hom}_{R^0\text{-Mod}}(\mathcal{M}, \mathcal{M}'_i) \arrow[r, "\sim"] & \mathrm{Hom}_{R^0\text{-Mod}}(\mathcal{M}, \mathcal{M}') \end{tikzcd} \tag{1.1}$$

where the lower horizontal arrow is bijective because $\mathrm{oblv}(\mathcal{M})$ is compact and oblv commutes with filtered colimits (being a left adjoint). On the other hand, the vertical arrows are injective because oblv is faithful. In particular, α is injective.

In order to show that α is surjective, we need show that any morphism $\varphi : \mathcal{M} \to \mathcal{M}'$ of $\mathcal{R}$-comodules comes from a morphism $\varphi_i : \mathcal{M} \to \mathcal{M}'_i$ of $\mathcal{R}$-comodules for some $i \in \mathcal{I}$. The commutative diagram (1.1) shows that $\mathrm{oblv}(\varphi)$ comes from a morphism $\nu : \mathcal{M} \to \mathcal{M}'_j$ *of R^0-modules.* In the following diagram, the outer rectangle and the right square commute:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\nu"] \arrow[d, "\rho_{\mathcal{M}}"] \arrow[dr, phantom, "(?)"] & \mathcal{M}'_j \arrow[r] \arrow[d, "\rho_{\mathcal{M}'_j}"] & \mathcal{M}' \arrow[d, "\rho_{\mathcal{M}'}"] \\ R^1 \underset{R^0}{\otimes} \mathcal{M} \arrow[r, "\mathrm{id}_{R^1} \otimes \nu"] & R^1 \underset{R^0}{\otimes} \mathcal{M}'_j \arrow[r] & R^1 \underset{R^0}{\otimes} \mathcal{M}' \end{tikzcd}$$

Thus, the two circuits of the left square $\rho_{\mathcal{M}'_j} \circ \nu$ and $(\mathrm{id}_{R^1} \otimes \nu) \circ \rho_{\mathcal{M}}$ are equalized after further mapping to $R^1 \underset{R^0}{\otimes} \mathcal{M}'$. Since the following canonical map of R^0-modules is a bijective ($\mathcal{I}_{j/} \to \mathcal{I}$ is cofinal):

$$\operatorname*{colim}_{i \in \mathcal{I}_{j/}} R^1 \underset{R^0}{\otimes} \mathcal{M}'_i \xrightarrow{\sim} R^1 \underset{R^0}{\otimes} \mathcal{M}',$$

we see that $\rho_{\mathcal{M}'_j} \circ \nu$ and $(\mathrm{id}_{R^1} \otimes \nu) \circ \rho_{\mathcal{M}}$ are equalized after mapping to $R^1 \underset{R^0}{\otimes} \mathcal{M}'_i$ along some arrow $j \to i$ in $\mathcal{I}$. Thus the composition $\mathcal{M} \xrightarrow{\nu} \mathcal{M}'_j \to \mathcal{M}'_i$ is a map of $\mathcal{R}$-comodules which gives rise to φ. □

1.2. **Representations "of finite rank".**

1.2.1. Let us now consider the analogue of finite-dimensional representations for affine category schemes.

For any k-algebra R, let $R\text{-Mod}^{\square}$ denote the full subcategory of R-Mod consisting of dualizable objects. One knows that $R\text{-Mod}^{\square}$ consists of finite projective R-modules. It has the structure of an exact category, when equipped with the tautological exact structure from R-Mod.

1.2.2. Suppose we are given a coalgebroid $\mathcal{R} = (R^0, R^1, \Delta, \varepsilon)$. Let $\mathcal{X} = (X^0, X^1, m, e)$ be the corresponding affine category scheme. Define:

$$\mathcal{X}\text{-Rep}^f := R^1\text{-Comod}(R^0\text{-Mod}^{\square}).$$

Write $F : \mathcal{X}\text{-Rep}^f \to R^0\text{-Mod}^{\square}$ for the forgetful functor. In other words, $\mathcal{X}\text{-Rep}^f$ is the full subcategory of $\mathcal{R}\text{-Comod} = R^1\text{-Comod}(R^0\text{-Mod})$ whose underlying R^0-module is dualizable.

Lemma 1.5. *The embedding $\mathcal{X}\text{-Rep}^f \hookrightarrow \mathcal{R}\text{-Comod}$ extends to a fully faithful embdding:*

$$\mathrm{Ind}(\mathcal{X}\text{-Rep}^f) \hookrightarrow \mathcal{R}\text{-Comod}. \tag{1.2}$$

Proof. One first notes that $\mathcal{R}$-Comod contains all colimits, in particular filtered ones. Thus $\mathcal{X}\text{-Rep}^f \hookrightarrow \mathcal{R}\text{-Comod}$ extends to a functor $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f) \to \mathcal{R}\text{-Comod}$. On the other hand, Lemma 1.4 shows that $\mathcal{X}\text{-Rep}^f$ belongs to the full subcategory of compact objects inside $\mathcal{R}$-Comod. Hence the extended functor remains fully faithful. □

1.2.3. The tensor product of R^1-comodules and the trivial R^1-comodule $\mathbf{1} := R^0$ equip the category $\mathcal{X}\text{-Rep}^f$ with the structure of a tensor category. With respect to the tautological tensor structure on $R^0\text{-Mod}^{\square}$, the forgetful functor F has the structure of a tensor functor.

Note that the embedding (1.2) is also a tensor functor, when $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$ is equipped with the ind-extended tensor structure.

1.2.4. Let us consider the functor:

$$\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F) : (R^0 \otimes R^0)\text{-ComAlg} \to \mathrm{Set}$$

which sends a commutative $(R^0 \otimes R^0)$-algebra R' to the set of morphisms of tensor functors $\mathrm{Hom}^{\otimes}(F \underset{R^0}{\otimes}{}_1 R', F \underset{R^0}{\otimes}{}_2 R')$, where the subscripts $i = 1, 2$ refer to the two R^0-structures of R'.

There is a canonical map of presheaves over $X^0 \times X^0$:

$$X^1 \to \mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F), \tag{1.3}$$

defined as follows. For a test affine scheme $S \to X^1$, we shall specify a map of tensor functors $F \underset{R^0}{\otimes} {}_1\mathcal{O}_S \to F \underset{R^0}{\otimes} {}_2\mathcal{O}_S$. For any $\mathcal{V} \in \mathcal{X}\text{-Rep}^f$, denote by $\underline{\mathcal{V}} := F(\mathcal{V})$ its underlying object in $R^0\text{-Mod}^\square$. The $\mathcal{R}$-comodule structure on $\underline{\mathcal{V}}$ corresponds to a map:

$$\underline{\mathcal{V}} \underset{R^0}{\otimes} {}_1R^1 \to \underline{\mathcal{V}} \underset{R^0}{\otimes} {}_2R^1.$$

Base change along $R^1 \to \mathcal{O}_S$ yields the desired map, obviously functorial in S.

1.3. Reconstruction of affine category schemes.

1.3.1. Let $\mathcal{X} = (X^0, X^1, m, e)$ be an affine category scheme. The goal of this subsection is to give a sufficient condition for the map (1.3) to be an isomorphism.

1.3.2. Recall the regular representation of §1.1.4, as an object $\mathrm{reg} \in \mathcal{R}\text{-Comod}$, whose underlying R^0-module is R^1 with respect to the left (i.e., pr_1) R^0-structure. Its comodule structure is given by the comultiplication map $\Delta : R^1 \to R^1 \underset{R^0}{\otimes} R^1$.

On the other hand, Lemma 1.5 tells us that it makes sense to say whether an object of $\mathcal{R}$-Comod belongs to $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$.

Lemma 1.6. *Suppose* $\mathrm{reg} \in \mathcal{R}\text{-Comod}$ *belongs to* $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$. *Then the canonical map of presheaves* (1.3) *is an isomorphism:*

$$X^1 \xrightarrow{\sim} \mathbf{Hom}^{\otimes}(\mathrm{pr}_1^*F, \mathrm{pr}_2^*F).$$

Proof. We explicitly construct the inverse of (1.3) in §1.3.3–1.3.8 below. ☐

1.3.3. *Observation.* We note that the hypothesis $\mathrm{reg} \in \mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$ implies that $\mathrm{Coind}(\mathcal{M}) \in \mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$ for any flat R^0-module $\mathcal{M}$. Indeed, any such $\mathcal{M}$ is a filtered colimit $\underset{\alpha}{\mathrm{colim}}(\mathcal{M}_\alpha)$ where $\mathcal{M}_\alpha$ is a free R^0-module of finite rank (Lazard's theorem). It suffices to verify the claim when $\mathcal{M}$ is free of finite rank. In this situation, $\mathrm{Coind}(\mathcal{M})$ is a finite direct sum of copies of reg, which belongs to $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$.

The hypothesis also implies that the underlying R^0-module $\underline{\mathrm{reg}}$ is flat, being a filtered colimit of finite projective R^0-modules.

1.3.4. Fix a test affine $S \to X^0 \times X^0$ together with a map of tensor functors from $\mathcal{X}\text{-Rep}^f$ to $\mathcal{O}_S\text{-Mod}^\square$:

$$\alpha : F \underset{R^0}{\otimes} {}_1\mathcal{O}_S \to F \underset{R^0}{\otimes} {}_2\mathcal{O}_S. \tag{1.4}$$

Regarding $F \underset{R^0}{\otimes} {}_1\mathcal{O}_S$ as a tensor functor $\mathcal{X}\text{-Rep}^f \to \mathcal{O}_S\text{-Mod}$, one obtains an ind-extension

$$\mathrm{Ind}(\mathcal{X}\text{-Rep}^f) \to \mathcal{O}_S\text{-Mod}, \quad \mathcal{V} \rightsquigarrow \underline{\mathcal{V}} \underset{R^0}{\otimes} {}_1\mathcal{O}_S,$$

where $\underline{\mathcal{V}}$ is the underlying R^0-module of $\mathcal{V}$, seen as an object of $\mathcal{R}$-Comod under the embedding of Lemma 1.5. Indeed, this expression of the ind-extended functor follows from the fact that both functors below preserve filtered colimits:

$$\mathcal{R}\text{-Comod} \xrightarrow{\mathrm{oblv}} R^0\text{-Mod} \xrightarrow{- \underset{R^0}{\otimes} {}_1\mathcal{O}_S} \mathcal{O}_S\text{-Mod}.$$

It is clear that an analogous statement holds for $F \underset{R^0}{\otimes} {}_2\mathcal{O}_S$.

By the functoriality of ind-extension, (1.4) induces a map between two tensor functors from $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$ to $\mathcal{O}_S$-Mod, again denoted by α. The hypothesis says that the regular representation belongs to $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$, so we may evaluate α on it to obtain a map of $\mathcal{O}_S$-modules:

$$\alpha_{\mathrm{reg}} : \mathcal{O}_S \underset{1,R^0}{\otimes} R^1 \to \mathcal{O}_S \underset{2,R^0}{\otimes} R^1. \tag{1.5}$$

(We exchange the positions to emphasize that the tensor uses the left R^0-structure of R^1.) One may also view α_{reg} as a map of R^0-modules $R^1 \to \mathcal{O}_S \underset{2,R^0}{\otimes} R^1$, where the R^0-module structure on $\mathcal{O}_S \underset{2,R^0}{\otimes} R^1$ comes from the first R^0-structure on $\mathcal{O}_S$.

1.3.5. We claim that α_{reg} (1.5) canonically upgrades to a map of (commutative) $\mathcal{O}_S$-algebras. Since α_{reg} is a map between tensor functors, it suffices to show that $\mathrm{reg} \in \mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$ is canonically a (commutative) algebra object. Indeed, the structure maps:

$$\mathrm{reg} \otimes \mathrm{reg} \to \mathrm{reg}, \quad \mathbf{1} \to \mathrm{reg}$$

are the ones induced from $R^1 \underset{1,R^0,1}{\otimes} R^1 \to R^1$ (multiplication) and $R^0 \to R^1$ (left R^0-structure), respectively. We omit checking that these maps are maps of comodules, where the coaction on $R^1 \underset{1,R^0,1}{\otimes} R^1$ is the diagonal one.

1.3.6. Having proved that (1.5) is a map of $\mathcal{O}_S$-algebras, we view it as a map between affine schemes:

$$\alpha_{\mathrm{reg}} : S \underset{2,X^0}{\times} X^1 \to X^1. \tag{1.6}$$

We claim that α_{reg} is $\mathcal{X}$-*equivariant*, in the sense that the following diagram commutes:

$$\begin{tikzcd} S \underset{2,X^0}{\times} X^1 \underset{X^0}{\times} X^1 \arrow[r, "\alpha_{\mathrm{reg}} \times \mathrm{id}_{X^1}"] \arrow[d, "\mathrm{id}_S \times m"] & X^1 \underset{X^0}{\times} X^1 \arrow[d, "\mathrm{id}_S \times m"] \\ S \underset{2,X^0}{\times} X^1 \arrow[r, "\alpha_{\mathrm{reg}}"] & X^1 \end{tikzcd} \tag{1.7}$$

where m denotes the composition law of the category scheme $\mathcal{X}$. To prove this statement, we first use the fact that the morphism in $\mathrm{Ind}(\mathcal{X}\text{-Rep}^f)$:

$$\mathrm{reg} \to \mathrm{Coind}(\underline{\mathrm{reg}}),$$

given by $\Delta : R^1 \to R^1{}_2 \underset{R^0}{\otimes} {}_1R^1$, is a map of algebra objects; then we appeal to the following result to identify $\alpha_{\mathrm{Coind}(\underline{\mathrm{reg}})}$ with $\alpha_{\mathrm{reg}} \times \mathrm{id}_{X^1}$ (applied to $\mathcal{N} := \underline{\mathrm{reg}}$).

Claim 1.7. For any flat R^0-module $\mathcal{N}$, there holds:

$$\alpha_{\mathrm{Coind}(\mathcal{N})} = \alpha_{\mathrm{reg}} \otimes \mathrm{id}_{\mathcal{N}} : R^1 \underset{R^0}{\otimes} \mathcal{N} \to \mathcal{O}_S \underset{2,R^0}{\otimes} R^1 \underset{R^0}{\otimes} \mathcal{N}$$

Proof. Since $\mathcal{N}$ is a filtered colimit of free R^0-modules of finite rank and $\alpha_{\mathcal{V}_1 \oplus \mathcal{V}_2} = \alpha_{\mathcal{V}_1} \oplus \alpha_{\mathcal{V}_2}$ for $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{X}\text{-Rep}^f$, the question reduces to the case $\mathcal{N} = R^0$, where it is obvious. $\square$

1.3.7. Next, we establish two statements.

(1) α_{reg} (1.6) is the unique $\mathcal{X}$-equivariant extension of its restriction to identity:

$$e^*\alpha_{\mathrm{reg}} : S \xrightarrow{\mathrm{id}\times e} S \underset{2,X^0}{\times} X^1 \xrightarrow{\alpha_{\mathrm{reg}}} X^1.$$

Namely, α_{reg} identifies with the composition:

$$S \underset{2,X^0}{\times} X^1 \xrightarrow{e^*\alpha_{\mathrm{reg}}\times \mathrm{id}_{X^1}} X^1 \underset{X^0}{\times} X^1 \xrightarrow{m} X^1.$$

(2) for any $\mathcal{V} \in \mathcal{X}\text{-}\mathrm{Rep}^f$, the map $\alpha_{\mathcal{V}}$ of $\mathcal{O}_S$-modules is the unique one making the following diagram commutes:

$$\begin{tikzcd} \underline{\mathcal{V}} \arrow[r, "\alpha_{\mathcal{V}}"] \arrow[d, "\rho_{\mathcal{V}}"] & \mathcal{O}_S \underset{2,R^0}{\otimes} \underline{\mathcal{V}} \arrow[d, "\mathrm{id}_S\otimes\rho_{\mathcal{V}}"] \\ R^1 \underset{R^0}{\otimes} \underline{\mathcal{V}} \arrow[r, "\alpha_{\mathrm{reg}}\otimes\mathrm{id}_{\underline{\mathcal{V}}}"] & \mathcal{O}_S \underset{2,R^0}{\otimes} R^1 \underset{R^0}{\otimes} \underline{\mathcal{V}} \end{tikzcd}$$

where $\rho_{\mathcal{V}}$ denotes the coaction map.

The first statement follows from restricting the commutative diagram (1.7) along the map:

$$S \underset{2,X^0}{\times} X^0 \underset{X^0}{\times} X^1 \xrightarrow{\mathrm{id}_S\times e\times \mathrm{id}_{X^1}} S \underset{2,X^0}{\times} X^1 \underset{X^0}{\times} X^1.$$

For the second statement, $\alpha_{\mathcal{V}}$ makes the diagram commute because $\rho_{\mathcal{V}} : \mathcal{V} \to \mathrm{Coind}(\underline{\mathcal{V}})$ is a morphism in $\mathrm{Ind}(\mathcal{X}\text{-}\mathrm{Rep}^f)$ and $\alpha_{\mathrm{Coind}(\underline{\mathcal{V}})} = \alpha_{\mathrm{reg}} \otimes \mathrm{id}_{\underline{\mathcal{V}}}$ by Claim 1.7. The map $\alpha_{\mathcal{V}}$ is the unique one with this property since both vertical arrows are injective, as demonstrated by the fact that the composition:

$$\underline{\mathcal{V}} \xrightarrow{\rho_{\mathcal{V}}} R^1 \underset{R^0}{\otimes} \underline{\mathcal{V}} \xrightarrow{\varepsilon\cdot\mathrm{id}_{\underline{\mathcal{V}}}} \underline{\mathcal{V}}$$

is the identity on $\underline{\mathcal{V}}$.

1.3.8. We define the map:

$$\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^*F, \mathrm{pr}_2^*F) \to X^1$$

of presheaves by sending an S-point α of $\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^*F, \mathrm{pr}_2^*F)$ to the S-point $e^*\alpha_{\mathrm{reg}}$.

It is clear that if α is the mapping of functors associated to some $g \in \mathrm{Maps}(S, X^1)$, then $e^*\alpha_{\mathrm{reg}}$ recovers g. Conversely, the map between tensor functors determined by $e^*\alpha_{\mathrm{reg}} \in \mathrm{Maps}(S, X^1)$ agrees with α, by the two statements established above. This completes the proof of Lemma 1.6.

2. Tannakian categories

We continue to fix the ground ring k. In this section, we discuss some constructions associated to the set-up of a tensor category $\mathcal{C}$, a commutative k-algebra R, and a tensor functor F valued in dualizable R-modules:

$$F : \mathcal{C} \to R\text{-Mod}^{\square}.$$

Write $X^0 := \mathrm{Spec}(R)$. Our first goal is to show that a certain mapping space of pullbacks of F is representable by an affine category scheme $\mathcal{X} = (X^0, X^1, m, e)$.

In the case where $\mathcal{C}$ is abelian and F is exact and faithful, we will identify the large category $\mathrm{Ind}(\mathcal{C})$ as comodules over $\mathcal{X}$.

2.1. Representabiliy of mapping spaces.

2.1.1. Let R be a commutative k-algebra. For any functor $F : \mathcal{C} \to R$-Mod and any commutative R-algebra R', we denote by:

$$F \otimes R' : \mathcal{C} \to R'\text{-Mod}$$

the functor which attaches to each $c \in \mathcal{C}$ the R'-module $F(c) \underset{R}{\otimes} R'$. Clearly, if F lands in $R\text{-Mod}^{\square}$, then $F \otimes R'$ lands in $R'\text{-Mod}^{\square}$.

When $\mathcal{C}$ has a symmetric monoidal structure and F is a symmetric monoidal functor, $F \otimes R'$ inherits a symmetric monoidal structure.

2.1.2. The following lemma is the basis of our representability statements.

Lemma 2.1. *Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a tensor category and R is a commutative k-algebra. Let*

$$F_1, F_2 : \mathcal{C} \to R\text{-Mod}^{\square}$$

be tensor functors. Then the presheaf:

$$\mathbf{Hom}^{\otimes}(F_1, F_2) : R\text{-ComAlg} \to \mathrm{Set}$$

which sends every commutative R-algebra R' to the set of morphisms of tensor functors $\mathrm{Hom}^{\otimes}(F_1 \otimes R', F_2 \otimes R')$ *is representable by an affine scheme over* $\mathrm{Spec}(R)$.

Proof. The proof is constructive and proceeds in several step.

Step 1. Consider the functor:

$$H : \mathrm{Tw}(\mathcal{C}) \to R\text{-Mod} \tag{2.1}$$

defined as follows:

(1) to $c \xrightarrow{\varphi} d$, it associates the R-module $F_1(c) \otimes F_2(d)^{\vee}$;
(2) to a morphism $\varphi \to \varphi'$ in $\mathrm{Tw}(\mathcal{C})$:

$$\begin{tikzcd} c \arrow[r, "\varphi"] \arrow[d] & d \\ c' \arrow[r, "\varphi'"] & d' \arrow[u] \end{tikzcd}$$

it associates the natural map:

$$F_1(c) \otimes F_2(d)^{\vee} \to F_1(c') \otimes F_2(d')^{\vee}.$$

Since R-Mod contains all colimits, we may set

$$\mathcal{A} := \underset{\mathrm{Tw}(\mathcal{C})}{\mathrm{colim}}\, H \in R\text{-Mod}.$$

Step 2. We will equip $\mathcal{A}$ with the structure of a commutative R-algebra. By Lemma A.10, it suffices to upgrade (2.1) to a symmetric monoidal functor, where the symmetric monoidal structure on $\mathrm{Tw}(\mathcal{C})$ is the one from §A.2.3.

The required data are specified by:

$$H(\mathbf{1} \xrightarrow{\varphi} \mathbf{1}) = F_1(\mathbf{1}) \otimes F_2(\mathbf{1})^\vee \xrightarrow{\sim} R,$$

and the commutative diagram:

$$\begin{tikzcd} H(c \xrightarrow{\varphi} d) \otimes H(c' \xrightarrow{\varphi'} d') \arrow[r] \arrow[d, equal] & H(c \otimes c' \xrightarrow{\varphi\otimes\varphi'} d \otimes d') \arrow[d, equal] \\ (F_1(c) \otimes F_2(d)^\vee) \otimes (F_1(c') \otimes F_2(d')^\vee) \arrow[r, "\sim"] & F_1(c \otimes c') \otimes F_2(d \otimes d')^\vee \end{tikzcd}$$

Step 3. For any commutative R-algebra R', we shall construct an isomorphism:

$$\mathrm{Hom}_{R\text{-Mod}}(\mathcal{A}, R') \xrightarrow{\sim} \mathrm{Hom}(F_1 \otimes R', F_2 \otimes R') \tag{2.2}$$

functorial in R', where Hom is calculated as plain (i.e., not symmetric monoidal) functors from $\mathcal{C}$ to R'-Mod.

Indeed, there are equivalences:

$$\begin{aligned} \mathrm{Hom}_{R\text{-Mod}}(\mathcal{A}, R') &\xrightarrow{\sim} \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Hom}_{R\text{-Mod}}(F_1(c) \otimes F_2(d)^\vee, R') \\ &\xrightarrow{\sim} \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Hom}_{R'\text{-Mod}}(F_1(c) \otimes R', F_2(d) \otimes R'), \end{aligned}$$

and the latter identifies with $\mathrm{Hom}(F_1 \otimes R', F_2 \otimes R')$ by Lemma A.8.

Step 4. We show that an R-module map $f : \mathcal{A} \to R'$ is a map of R-algebras if and only if its image α under (2.2) is compatible with the symmetric monoidal structures on $F_1 \otimes R'$ and $F_2 \otimes R'$. For any object $c \in \mathcal{C}$, we let α_c denote the component:

$$\alpha_c : F_1(c) \to F_2(c) \otimes R'.$$

We consider the unit and product structures seperately.

(1) The statement that f restricts to the given map $R \to R'$ is equivalent to the commutativity of the the diagram:

$$\begin{tikzcd} R \arrow[r, "\sim"] \arrow[dr] & F_1(\mathbf{1}) \arrow[d, "\alpha_{\mathbf{1}}"] \\ & F_2(\mathbf{1}) \otimes R' \end{tikzcd}$$

which is in turn equivalent to the compatibility between α and the unital structures on $F_1 \otimes R'$ and $F_2 \otimes R'$.

(2) The statement that f preserves products is equivalent to the commutativity of the diagram for all $\varphi : c \to d$ and $\varphi' : c' \to d'$:

$$\begin{tikzcd}
(F_1(c) \otimes F_2(d)^\vee) \otimes (F_1(c') \otimes F_2(d')^\vee) \arrow[r, "\sim"] \arrow[d, "f\otimes f"] & F_1(c \otimes c') \otimes F_2(d \otimes d')^\vee \arrow[d, "f"] \\
R' \otimes R' \arrow[r] & R'
\end{tikzcd}$$

where we slightly abused the notation f for its restriction to each term of the form $F_1(c) \otimes F_2(d)^\vee$. The functoriality of the upper horizontal isomorphism in φ, φ' allows to reduce this statement to the special case $\varphi = \mathrm{id}_c$ and $\varphi' = \mathrm{id}_{c'}$. This latter statement is equivalent to the commutativity of the square:

$$\begin{tikzcd}
F_1(c) \otimes F_1(c') \arrow[r] \arrow[d, "\alpha_c \otimes \alpha_{c'}"] & F_1(c \otimes c') \arrow[d, "\alpha_{c\otimes c'}"] \\
(F_2(c) \otimes R') \otimes (F_2(c') \otimes R') \arrow[r] & F_2(c \otimes c') \otimes R'
\end{tikzcd}$$

which expresses the compatibility of α with monoidal products on $F_1 \otimes R'$ and $F_2 \otimes R'$.

Therefore, the isomorphism (2.2) restricts to an isomorphism:

$$\mathrm{Hom}_{R\text{-ComAlg}}(\mathcal{A}, R') \xrightarrow{\sim} \mathrm{Hom}^{\otimes}(F_1 \otimes R', F_2 \otimes R'), \tag{2.3}$$

which is functorial in R'. $\square$

2.2. Obtaining affine category schemes.

2.2.1. In this subsection, we will obtain from a tensor category $\mathcal{C}$ and a tensor functor $F : \mathcal{C} \to R\text{-Mod}^{\square}$ an affine category scheme $\mathcal{X}$. We begin with the following representability result of the scheme of "arrows."

Lemma 2.2. *Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a tensor category and R is a commutative k-algebra. Let:*

$$F : \mathcal{C} \to R\text{-Mod}^{\square}$$

be a tensor functor. Then the presheaf:

$$\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F) : (R \otimes R)\text{-ComAlg} \to \mathrm{Set}$$

which sends a commutative $(R \otimes R)$-algebra R' to the set of morphisms of tensor functors $\mathrm{Hom}^{\otimes}(F \underset{R}{\otimes}{}_1 R', F \underset{R}{\otimes}{}_2 R')$, where the subscripts $i = 1, 2$ refer to the two R-structures of R', is representable by an affine scheme over $\mathrm{Spec}(R \otimes R)$.

Proof. We write $F_1 : \mathcal{C} \to (R \otimes R)\text{-Mod}^{\text{dualizable}}$ for the functor:

$$\mathrm{pr}_1^* F : c \rightsquigarrow F(c) \underset{R}{\otimes}{}_1 (R \otimes R) \cong F(c) \underset{k}{\otimes} R,$$

and $F_2 : \mathcal{C} \to (R \otimes R)\text{-Mod}^{\text{dualizable}}$ for the functor:

$$\mathrm{pr}_2^* F : c \rightsquigarrow F(c) \underset{R}{\otimes}{}_2 (R \otimes R) \cong R \underset{k}{\otimes} F(c).$$

Then we apply Lemma 2.1 to F_1 and F_2. $\square$

In particular, the output of the construction of Lemma 2.1 gives the representing commutative $(R \otimes R)$-algebra:

$$\mathcal{A} \cong \underset{(c \to d) \in \mathrm{Tw}(\mathcal{C})}{\mathrm{colim}} F(c) \underset{k}{\otimes} F(d)^{\vee}, \tag{2.4}$$

where the dual is taken in $R\text{-Mod}^{\square}$.

2.2.2. Let us remain in the set-up of Lemma 2.2. Denote by $X^0 := \mathrm{Spec}(R)$ and $X^1 := \mathrm{Spec}(\mathcal{A})$ where $\mathcal{A}$ is the representing commutative $(R \otimes R)$-algebra. We denote the two projection maps, corresponding to the two R-structures of $\mathcal{A}$, by:

$$X^1 \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X^0.$$

Our current goal is to equip this diagram with the structure of a category scheme. This involves specifying the additional data of composition and identity:

$$X^1{}_{\mathrm{pr}_2}\underset{X^0}{\times}{}_{\mathrm{pr}_1} X^1 \to X^1 \tag{2.5}$$

$$X^0 \to X^1 \tag{2.6}$$

More precisely, (2.5) arises from the map of functors $(R \otimes R \otimes R)\text{-ComAlg} \to \mathrm{Set}$:

$$\circ : \mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F) \times \mathbf{Hom}^{\otimes}(\mathrm{pr}_2^* F, \mathrm{pr}_3^* F) \to \mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_3^* F),$$

and (2.6) arises from the map $\mathcal{A} \to R$ of commutative $(R \otimes R)$-algebras, where R is given the diagonal $(R \otimes R)$-action, represented by:

$$\mathrm{id} \in \mathrm{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F), \quad \mathrm{pr}_1^* F \cong \mathrm{pr}_2^* F.$$

2.2.3. In algebraic terms, the two maps above give rise to k-algebra morphisms

$$\mathcal{A} \to \mathcal{A}_2 \underset{R}{\otimes} {}_1\mathcal{A} \tag{2.7}$$

$$\mathcal{A} \to R \tag{2.8}$$

The first map is $(R \otimes R)$-linear, where the two R's act on $\mathcal{A}_2 \underset{R}{\otimes} {}_1\mathcal{A}$ by outer multiplications. (We think of 1 as the left R-structure and 2 as the right R-structure.) The second map is also $(R \otimes R)$-linear, where R is equipped with the diagonal action.

Regarding $\mathcal{A}$ as an (R, R)-bimodule, the endomorphism:

$$\mathcal{A} \underset{R}{\otimes} (-) : R\text{-Mod} \to R\text{-Mod} \tag{2.9}$$

upgrades to a comonad acting on R-Mod, with comultiplication given by (2.7) and counit given by (2.8). This is the comonad associated to the forgetful–coinduction adjunction (see §1.1.4).

2.3. **The ind-category as comodules.**

2.3.1. *Set-up.* Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a tensor category *containing finite colimits*. Suppose R is a commutative k-algebra and

$$F : \mathcal{C} \to R\text{-Mod}$$

is a tensor functor preserving finite colimits.

Thus, $\mathrm{Ind}(\mathcal{C})$ contains all colimits and the ind-extension $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \to R\text{-Mod}$ preserves all colimits (Lemma A.12). By the adjoint functor theorem, it admits a right adjoint G:

$$\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \rightleftarrows R\text{-Mod} : G.$$

We let T denote the comonad $\mathrm{Ind}(F) \circ G$ acting on R-Mod. In particular, $\mathrm{Ind}(F)$ canonically factors as follows, where oblv denotes the forgetful functor.

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{C}) \arrow[r, "\widetilde{F}"] \arrow[dr, "\mathrm{Ind}(F)"'] & T\text{-Comod}(R\text{-Mod}) \arrow[d, "\mathrm{oblv}"] \\ & R\text{-Mod} \end{tikzcd}$$

2.3.2. The following is a consequence of the Barr–Beck theorem.

Lemma 2.3. *In set-up §2.3.1, if $\mathcal{C}$ is abelian and $F : \mathcal{C} \to R\text{-Mod}$ is exact and faithful. Then the functor $\widetilde{F}$ is an equivalence:*

$$\widetilde{F} : \mathrm{Ind}(\mathcal{C}) \xrightarrow{\sim} T\text{-Comod}(R\text{-Mod}).$$

Proof. Since filtered colimits in R-Mod are exact, the ind-extension:

$$F : \mathrm{Ind}(\mathcal{C}) \to R\text{-Mod} \tag{2.10}$$

is still an exact and faithful functor between abelian categories. We observe:

(1) The functor (2.10) is conservative. Indeed, if a morphism $\varphi : c \to d$ in $\mathrm{Ind}(\mathcal{C})$ becomes an isomorphism after applying F, the exactness of F implies that $F(\mathrm{Ker}(\varphi)) = 0$ and $F(\mathrm{Coker}(\varphi)) = 0$. The faithfulness of F then implies that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are zero, so φ is an isomorphism.
(2) $\mathrm{Ind}(\mathcal{C})$ admits cosplit equalizers, and the functor (2.10) preserves them. Indeed, any abelian category contains finite limits, and the functor (2.10) preserves them because it is exact.

The result thus follows from the Barr–Beck theorem (Lemma A.9). ∎

2.4. Identification of the comonad.

2.4.1. Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a tensor category and R is a commutative k-algebra. Let:

$$F : \mathcal{C} \to R\text{-Mod}^{\square}$$

be a tensor functor. The materials from §2.2 yield an affine scheme $\mathrm{Spec}(\mathcal{A})$ representing $\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F)$, and moreover a comonad $\mathcal{A} \underset{R}{\otimes} (-)$ (2.9) acting on R-Mod.

On the other hand, if $\mathcal{C}$ contains finite colimits and the induced functor $\mathcal{C} \to R\text{-Mod}$ (denoted by the same letter F) preserves them, then the ind-extension together with its right adjoint define a comonad $T := \mathrm{Ind}(F) \circ G$ acting on R-Mod as in §2.3.1.

2.4.2. Note that there is a morphism of endo-functors on R-Mod:

$$\mathrm{Ind}(F)\circ G\to \mathcal{A}\underset{R}{\otimes}(-). \tag{2.11}$$

Indeed, given any R-module $\mathcal{N}$. The unit $R\to\mathcal{A}$ of the right R-module structure gives rise to a map $\mathcal{N}\to\mathcal{A}\underset{R}{\otimes}\mathcal{N}$. Upon applying G, we obtain a map $G(\mathcal{N})\to G(\mathcal{A}\underset{R}{\otimes}\mathcal{N})$ in $\mathrm{Ind}(\mathcal{C})$, and finally (2.11) comes from adjunction.

Lemma 2.4. *The map* (2.11) *is an isomorphism of comonads.*

Proof. Let us first prove that for R-modules $\mathcal{N},\mathcal{N}'$, the canonical map defined above:

$$\mathrm{Hom}_{R\text{-Mod}}(\mathcal{A}\underset{R}{\otimes}\mathcal{N},\mathcal{N}')\to\mathrm{Hom}_{R\text{-Mod}}(\mathrm{Ind}(F)\circ G(\mathcal{N}),\mathcal{N}') \tag{2.12}$$

is an isomorphism. The right-hand-side identifies with $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(G(\mathcal{N}),G(\mathcal{N}'))$ by adjunction. To identify the left-hand-side, we recall the structure of $\mathcal{A}$ as an (R,R)-bimodule (2.4), which yields:

$$\mathrm{Hom}_{R\text{-Mod}}(\mathcal{A}\underset{R}{\otimes}\mathcal{N},\mathcal{N}')\cong\lim_{(c\to d)\in\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}}\mathrm{Hom}_{R\text{-Mod}}((F(c)\underset{k}{\otimes}F(d)^\vee)\underset{R}{\otimes}\mathcal{N},\mathcal{N}').$$

(To interpret the right-hand-side: the tensor with $\mathcal{N}$ is formed with respect to the R-module structure on $F(c)\underset{k}{\otimes}F(d)^\vee$ acting on $F(d)^\vee$. The remaining R-module structure on:

$$(F(c)\underset{k}{\otimes}F(d)^\vee)\underset{R}{\otimes}\mathcal{N}\cong F(c)\underset{k}{\otimes}(F(d)^\vee\underset{R}{\otimes}\mathcal{N})$$

acts only on the factor $F(c)$, which we then use to form the hom-space with $\mathcal{N}'$.) Consequently, we have

$$\begin{aligned}
&\lim_{(c\to d)\in\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}}\mathrm{Hom}_{R\text{-Mod}}((F(c)\underset{k}{\otimes}F(d)^\vee)\underset{R}{\otimes}\mathcal{N},\mathcal{N}')\\
\xrightarrow{\sim}&\lim_{(c\to d)\in\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}}\mathrm{Hom}_{k\text{-Mod}}(F(d)^\vee\underset{R}{\otimes}\mathcal{N},F(c)^\vee\underset{R}{\otimes}\mathcal{N}')\\
\xrightarrow{\sim}&\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},k\text{-Mod})}(F(-)^\vee\underset{R}{\otimes}\mathcal{N},F(-)^\vee\underset{R}{\otimes}\mathcal{N}').
\end{aligned}\tag{2.13}$$

where we applied Lemma A.8 in the final step. On the other hand, the functor:

$$F(-)^\vee\underset{R}{\otimes}\mathcal{N}:\mathcal{C}^{\mathrm{op}}\to k\text{-Mod}$$

is ind-represented by $G(\mathcal{N})$. Namely, there is a canonical isomorphism of k-modules, functorial in $c\in\mathcal{C}$:

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(c,G(\mathcal{N}))\xrightarrow{\sim}\mathrm{Hom}_{R\text{-Mod}}(F(c),\mathcal{N})\xrightarrow{\sim}F(c)^\vee\underset{R}{\otimes}\mathcal{N}$$

Therefore, the hom-space of functors (2.13) is equivalent to $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(G(\mathcal{N}),G(\mathcal{N}'))$. This shows that (2.12) is an isomorphism.

It remains to prove that this morphism is compatible with the monad structures. Namely, the following diagrams commute:

$$\begin{tikzcd}
\mathrm{Ind}(F)\circ G \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\cdot\mathrm{unit}\cdot\mathrm{id}"] & \mathcal{A}\underset{R}{\otimes}(-) \arrow[d, "\Delta\otimes\mathrm{id}"] \\
\mathrm{Ind}(F)\circ G\circ\mathrm{Ind}(F)\circ G \arrow[r, "\sim"] & \mathcal{A}\underset{R}{\otimes}\mathcal{A}\underset{R}{\otimes}(-)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Ind}(F)\circ G \arrow[r, "\sim"] \arrow[d, "\mathrm{counit}"] & \mathcal{A}\underset{R}{\otimes}(-) \arrow[d, "\varepsilon"] \\
\mathrm{id} \arrow[r, "\sim"] & \mathrm{id}
\end{tikzcd}$$

We leave these verifications to the interested reader. $\square$

2.5. **Reconstruction of the ind-category.**

2.5.1. We will now put together the results from this section to prove a reconstruction statement of the ind-category. Let us assume:

(1) $\mathcal{C}$ is an *abelian* tensor category;
(2) R is a commutative k-algebra;
(3) $F : \mathcal{C} \to R\text{-Mod}^{\square}$ is an *exact*, *faithful* tensor functor.

Let $\mathcal{X} = (R, \mathcal{A}, \Delta, \varepsilon)$ be the coalgebroid representing $\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F)$, constructed in §2.2. Let $\mathcal{A}$-Comod denote the tensor category of $\mathcal{A}$-comodules (in R-Mod).

Lemma 2.5. *With the above notations, there is a canonical equivalence:*

$$\widetilde{F} : \mathrm{Ind}(\mathcal{C}) \xrightarrow{\sim} \mathcal{A}\text{-Comod},$$

making the following diagram commute:

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{C}) \arrow[r, "\widetilde{F}"] \arrow[dr, "\mathrm{Ind}\, F"'] & \mathcal{A}\text{-Comod} \arrow[d, "\mathrm{oblv}"] \\ & R\text{-Mod} \end{tikzcd}$$

Proof. We first apply Lemma 2.3 to obtain an equivalence:

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{\sim} T\text{-Comod}(R\text{-Mod}),$$

where T is the comonad $\mathrm{Ind}(F) \circ G$ (see §2.3.1). By Lemma 2.4, the comonad T identifes with tensoring with the co-algebra $\mathcal{A}$ representing $\mathbf{Hom}^{\otimes}(\mathrm{pr}_1^* F, \mathrm{pr}_2^* F)$. $\square$

3. Recognizing small Tannakian categories

3.1. **Ramblings.** This section is not yet written. In the previous section, we showed that for a tensor category $\mathcal{C}$ with a tensor functor:

$$F : \mathcal{C} \to R\text{-Mod}^{\square}$$

satisfying additional conditions, such as $\mathcal{C}$ being abelian and F exact and faithful, one can recover $\mathrm{Ind}(\mathcal{C})$ as $\mathcal{A}$-comodules in R-Mod. Here, $X^0 = \mathrm{Spec}(R)$ and $X^1 = \mathrm{Spec}(\mathcal{A})$ is the representing affine category scheme $\mathcal{X}$.

Given the additional assumption that $\mathcal{C}$ is rigid, one can show that $\mathcal{X}$ is a groupoid scheme. Assuming further that k is a field, one can recover the small category $\mathcal{C}$ as $\mathcal{A}$-comodules in $R\text{-Mod}^{\square}$ using the following approach:

(1) prove that $X^1 \to X^0 \times X^0$ is faithfully flat, i.e., the groupoid is transitive. In [Deligne, Catégories Tannakiennes], this is done using a construction of tensor products of Tannakian categories. Namely, one regards $\mathcal{A} = \operatorname{colim} F(c) \otimes F(d)^{\vee}$ as the image of an *ind-object* under the fiber functor:

$$F \otimes F : \mathcal{C} \otimes \mathcal{C} \to (R \otimes R)\text{-Mod}^{\square}$$

This ind-object of $\mathcal{C} \otimes \mathcal{C}$, which is *a priori* $\operatorname{colim} c \boxtimes d^{\vee}$, can be expressed as a filtered colimit of quotients of $\bigoplus_{c \in C_i} c \boxtimes c^{\vee}$ as one expresses $\mathcal{C}$ as a filtered colimit of its finite full subcategories $\mathcal{C}_i$.

(2) one uses the trick that representations of a transitive groupoid (over a field) are equivalent to representations of the stabilzer group of a field-valued point. Thus, one obtains the desired equivalence by taking compact objects on both sides of:

$$\mathrm{Ind}(\mathcal{C}) \cong \mathcal{A}\text{-Comod}.$$

I am not yet sure that this is the generality that makes most sense. In particular, it assumes $\mathcal{C}$ to be abelian (so $\mathrm{Vect}(\mathbb{A}^1/\mathbb{G}_m)$ is excluded) and rigid (so representations of monoids are excluded), and the assumption that k is a field appears *ad hoc*. (On the other hand, in order to find a more general statement, it seems that one should first think about reconstruction of large categories with the abelian-ness assumption removed, so I am delaying writing up this section.)

Appendix A. Category theory lemmas

A.1. Recovering a category from its ind-completion.

A.1.1. The following discussion applies to categories in general as well as k-linear categories for a fixed ring k. Let $\mathcal{C}$ be a category (resp. a k-linear category).

An object $c \in \mathcal{C}$ is called *compact* if for every filtered diagram $d_i \in \mathcal{C}$ ($i \in \mathcal{I}$) whose colimit exists, the natural map below is an isomorphism:

$$\operatorname*{colim}_{i\in\mathcal{I}} \operatorname{Hom}(c, d_i) \to \operatorname{Hom}(c, \operatorname*{colim}_{i\in\mathcal{I}} d_i).$$

Let $\mathcal{C}^{\mathrm{cpt}}$ denote the full subcategory of $\mathcal{C}$ spanned by compact objects.

A.1.2. Let $c \in \mathcal{C}$ be any object. A morphism $e : c \to c$ is called *idempotent* if $e^2 = e$. A *splitting* of an idempotent $e : c \to c$ is the datum (d, ι, r) where $d \in \mathcal{C}$ is an object and ι, r are morphisms:

$$c \xrightarrow{r} d \xrightarrow{\iota} c, \quad \text{such that } r \circ \iota = \mathrm{id}_d \text{ and } \iota \circ r = e.$$

A splitting of e, if exists, is unique up to a unique isomorphism. We call the resulting object d the *image* of e.

Lemma A.1. *Let $e : c \to c$ be an idempotent. The following are equivalent:*

(1) e splits;
(2) the diagram $c \xrightarrow{e} c$ admits a limit;
(3) the diagram $c \xrightarrow{e} c$ admits a colimit.

When these conditions hold, the image of e identifies with both the limit and colimit of the diagram $c \xrightarrow{e} c$. □

A.1.3. We call $\mathcal{C}$ *idempotent-complete* (or *Karoubian*) if every idempotent in $\mathcal{C}$ splits.[1] By Lemma A.1, a category $\mathcal{C}$ which admits finite limits *or* finite colimits is idempotent-complete. This yields:

Lemma A.2. *An abelian category $\mathcal{C}$ is idempotent-complete.* □

A.1.4. Let $\mathrm{Ind}(\mathcal{C})$ denote the ind-completion of $\mathcal{C}$. We view $\mathcal{C}$ as a full subcategory of $\mathrm{Ind}(\mathcal{C})$ via the tautological embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$. By definition of $\mathrm{Ind}(\mathcal{C})$, this embedding factors through:

$$\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})^{\mathrm{cpt}} \hookrightarrow \mathrm{Ind}(\mathcal{C}).$$

Lemma A.3. *Suppose $d \in \mathrm{Ind}(\mathcal{C})$. The following are equivalent.*

(1) $d \in \mathrm{Ind}(\mathcal{C})^{\mathrm{cpt}}$;
(2) there is an idempotent $e : c \to c$ in $\mathcal{C}$ such that d is isomorphic to the image of e.

---

[1] This notion is discussed in [09SF] in the context of pre-additive categories.

(2) $\Longrightarrow$ (1). Suppose $e : c \to c$ is an idempotent in $\mathcal{C}$ with image d. For any filtered diagram b_i in $\mathrm{Ind}(\mathcal{C})$ ($i \in \mathcal{I}$) with colimit b, the following diagram commutes:

$$\begin{tikzcd}
\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}(d.b_i) \arrow[r, "\alpha_d"] \arrow[d] & \mathrm{Hom}(d,b) \arrow[d] \\
\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}(c.b_i) \arrow[r, "\alpha_c"] \arrow[d] & \mathrm{Hom}(c,b) \arrow[d] \\
\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}(d.b_i) \arrow[r, "\alpha_d"] & \mathrm{Hom}(d,b)
\end{tikzcd}$$

where α_c (resp. α_d) is the tautological map and both vertical compositions are identity. Since α_c is a bijection, the top commutative square shows that α_d is injective and the bottom commutative square shows that α_d is surjective. Hence α_d is a bijection as well. □

(1) $\Longrightarrow$ (2). Let d be a compact object of $\mathrm{Ind}(\mathcal{C})$. By definition of $\mathrm{Ind}(\mathcal{C})$, there is a filtered system d_i in $\mathcal{C}$ ($i \in \mathcal{I}$) with $d \cong \operatorname*{colim}_{i\in\mathcal{I}} d_i$, where the colimit is taken in $\mathrm{Ind}(\mathcal{C})$. Since d is compact, the natural map below is bijective:

$$\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}(d, d_i) \to \mathrm{Hom}(d,d),$$

so the morphism id_d factors as:

$$d \xrightarrow{\iota} d_i \xrightarrow{r} d$$

for some $i \in \mathcal{I}$. In particular, $e := \iota \circ r$ is an idempotent acting on $d_i \in \mathcal{C}$. This identifies d as the image of a split idempotent in $\mathcal{C}$. □

Lemma A.4. *Suppose $\mathcal{C}$ is idempotent-complete. Then the canonical embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})^{\mathrm{cpt}}$ is an equivalence.*

Proof. This follows from Lemma A.3. □

Lemma A.5. *Suppose $\mathcal{C}$ is abelian. Then the canonical embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})^{\mathrm{cpt}}$ is an equivalence.*

Proof. This follows form Lemma A.2 and Lemma A.4. □

A.1.5. Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal category.

Lemma A.6. *Suppose $\mathbf{1} \in \mathcal{C}$ is compact and tensoring with any object $- \otimes c : \mathcal{C} \to \mathcal{C}$ preserves filtered colimits. Then any dualizable object of $\mathcal{C}$ is compact.*

Proof. Let $c \in \mathcal{C}$ be a dualizable object. Consider any filtered system $d_i \in \mathcal{C}$ ($i \in \mathcal{I}$) whose colimit exists, and is denoted by d. We have a commutative diagram:

$$\begin{tikzcd}
\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}(c, d_i) \arrow[rr] \arrow[d, "\cong"] & & \mathrm{Hom}(c,d) \arrow[d, "\cong"] \\
\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}(\mathbf{1}, c^\vee \otimes d_i) \arrow[r] & \mathrm{Hom}(\mathbf{1}, \operatorname*{colim}_{i\in\mathcal{I}} (c^\vee \otimes d_i)) \arrow[r] & \mathrm{Hom}(\mathbf{1}, c^\vee \otimes d)
\end{tikzcd}$$

The two lower horizontal arrows are bijective, because $\mathbf{1}$ is compact and tensoring with $c^\vee$ preserves filtered colimits. Thus the upper horizontal arrow is bijective. □

Lemma A.7. *Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal abelian category in which every $c \in \mathcal{C}$ is dualizable. Then the canonical embedding of symmetric monoidal categories*

$$(\mathcal{C}, \otimes, \mathbf{1}) \hookrightarrow (\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}\otimes, \mathbf{1})$$

identifies $\mathcal{C}$ with the full subcategory of $\mathrm{Ind}(\mathcal{C})$ consisting of dualizable objects.

Proof. Since $\mathcal{C}$ is abelian, the embedding $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ identifies $\mathcal{C}$ with the compact objects in $\mathrm{Ind}(\mathcal{C})$ by Lemma A.5. Since every object in $\mathcal{C}$ is dualizable (as an object of $\mathcal{C}$, hence also as an object of $\mathrm{Ind}(\mathcal{C})$), we have a containement:

$$\mathrm{Ind}(\mathcal{C})^{\mathrm{cpt}} \subset \mathrm{Ind}(\mathcal{C})^{\mathrm{dualizable}} \tag{A.1}$$

On the other hand, $\mathbf{1}$ is compact in $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Ind}\otimes$ preserves filtered colimits [Amelie, §1.4], so we may apply Lemma A.6 to $\mathrm{Ind}(\mathcal{C})$ and conclude that (A.1) is an equivalence. □

A.2. Twisted arrows.

A.2.1. Let $\mathcal{C}$ be a category (resp. a k-linear category for a fixed ring k.) Write $\mathrm{Tw}(\mathcal{C})$ for the category whose objects are morphisms $\varphi : c \to d$ in $\mathcal{C}$, and a morphism $\varphi \to \varphi'$ is given by a commutative diagram:

$$\begin{tikzcd} c \arrow[r, "\varphi"] \arrow[d] & d \\ c' \arrow[r, "\varphi'"] & d' \arrow[u] \end{tikzcd}$$

Clearly, a morphism $\varphi \to \varphi'$ in $\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}$ is a commutative diagram:

$$\begin{tikzcd} c \arrow[r, "\varphi"] & d \arrow[d] \\ c' \arrow[r, "\varphi'"] \arrow[u] & d' \end{tikzcd}$$

A.2.2. The twisted arrow category allows one to express morphisms between functors as a suitable limit.

Lemma A.8. *Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are functors. Then there is a natural bijection:*

$$\mathrm{Hom}(F, G) \xrightarrow{\sim} \lim_{(c \to d) \in \mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}}(F(c), G(d)).$$

A.2.3. Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal category. Then $\mathrm{Tw}(\mathcal{C})$ inherits a symmetric monoidal structure defined as follows. For any $\varphi : c \to d$, $\varphi' : c' \to d'$, we write $\varphi \otimes \varphi'$ for the monoidal product:

$$\varphi \otimes \varphi' : c \otimes c' \to d \otimes d'.$$

This product is clearly functorial in φ and φ'. The unit in $\mathrm{Tw}(\mathcal{C})$ is define to be the identity arrow on $\mathbf{1} \in \mathcal{C}$. The natural isomorphisms:

$$\mathbf{1} \otimes - \xrightarrow{\sim} \mathrm{id}_{\mathrm{Tw}(\mathcal{C})}, \quad - \otimes \mathbf{1} \xrightarrow{\sim} \mathrm{id}_{\mathrm{Tw}(\mathcal{C})}$$

are the obvious ones.

The associativity constraint is inherited from $\mathcal{C}$ as the commutative diagram:

$$\begin{tikzcd} (c \otimes c') \otimes c'' \arrow[r, "(\varphi\otimes\varphi')\otimes\varphi''"] \arrow[d, "\cong"] & (d \otimes d') \otimes d'' \\ c \otimes (c' \otimes c'') \arrow[r, "\varphi\otimes(\varphi'\otimes\varphi'')"] & d \otimes (d' \otimes d'') \arrow[u, "\cong"'] \end{tikzcd}$$

and the commutativity constraint is inherited from $\mathcal{C}$ as the commutative diagram:

$$\begin{tikzcd} c \otimes c' \arrow[r, "\varphi'\otimes\varphi"] \arrow[d, "\cong"] & d \otimes d' \\ c' \otimes c \arrow[r, "\varphi\otimes\varphi'"] & d' \otimes d \arrow[u, "\cong"'] \end{tikzcd}$$

We omit verifying the coherence conditions satisfied by these data.

A.3. **(Co)monadicity.**

A.3.1. Let $\mathcal{C}$ be a category (resp. k-linera category). A *comonad* acting on $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ equipped with the additional data:

(1) a comultiplication $\Delta : T \to T \circ T$;
(2) a counit $\varepsilon : T \to \mathrm{id}_{\mathcal{C}}$,

satisfying the coassociative and counital conditions. In other words, a comonad is a coassciative coalgebra in the monoidal category $(\mathrm{End}(\mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$.

A.3.2. Suppose (T, Δ, ε) is a comonad acting on $\mathcal{C}$. We slightly abuse the notation and refer to the endofunctor T as the comonad. A *T-comodule* in $\mathcal{C}$ is an object $c \in \mathcal{C}$ equipped with a coaction:

$$\alpha : c \to T(c),$$

satisfying the following two conditions:

(1) the two compositions are equal:

$$\begin{tikzcd} c \arrow[r, "\alpha"] & T(c) \arrow[r, shift left, "\Delta"] \arrow[r, shift right, "T(\alpha)"'] & T \circ T(c). \end{tikzcd}$$

(2) the composition $c \xrightarrow{\alpha} T(c) \xrightarrow{\varepsilon} c$ is the identity on c.

In other words, a T-comodule in $\mathcal{C}$ is a comodule object in $\mathcal{C}$ with respect to the canonical $\mathrm{End}(\mathcal{C})$-action on $\mathcal{C}$ and the coassociative coalgebra object $T \in \mathrm{End}(\mathcal{C})$.

A.3.3. *Set-up.* Suppose we are given an adjuction:

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

Then the endofunctor $T := F \circ G$ has a canonical structure of a comonad acting on $\mathcal{D}$. Furthermore, the functor F canonically factors as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & T\text{-Comod}(\mathcal{D}) \arrow[d, "\mathrm{oblv}"] \\ & \mathcal{D} \end{tikzcd}$$

where oblv is the forgetful functor.

A.3.4. We call a functor $F : \mathcal{C} \to \mathcal{D}$ *conservative* if for every morphism $\varphi : c_1 \to c_2$ in $\mathcal{C}$, if $F(\varphi)$ is an isomorphism, then so is φ. The following is a form of the Barr–Beck theorem.

Lemma A.9. *In the set-up §A.3.3, the functor $\widetilde{F}$ is an equivalence if and only if the following conditions are satisfied.*

(1) F is conservative;
(2) $\mathcal{C}$ contains F-cosplit equalizers and F preserves them.

A.4. Miscellaneous facts.

A.4.1. A fact about obtaining commutative algebras:

Lemma A.10. *Suppose $(\mathcal{C}, \otimes, \mathbf{1})$ is a symmetric monoidal category, in which $\otimes$ commutes with all colimits. Let $F : (\mathcal{I}, \otimes, \mathbf{1}) \to (\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal functor. Then the natural maps:*

$$\mathbf{1} \xrightarrow{\sim} F(\mathbf{1})$$

$$F(i) \otimes F(j) \xrightarrow{\sim} F(i \otimes j)$$

define a commutative algebra structure on $\operatorname{colim}_{\mathcal{I}} F \in \mathcal{C}$. □

A.4.2. Let $\mathcal{C}$ be a category (resp. a k-linear category).

Lemma A.11. *The canonical embedding $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$ preserves finite colimits.*

Proof. This follows from the fact that in Set (resp. k-Mod), finite limits commute with filtered colimits. □

Lemma A.12. *Suppose $\mathcal{C}$ contains finite colimits. Then*

(1) $\operatorname{Ind}(\mathcal{C})$ contains all colimits.
(2) For any functor $F : \mathcal{C} \to \mathcal{D}$ preserving finite colimits, where $\mathcal{D}$ is a cocomplete category, the ind-extension

$$\operatorname{Ind}(F) : \operatorname{Ind}(\mathcal{C}) \to \mathcal{D}$$

preserves all colimits.

Proof. This follows from Lemma A.11 together with the fact that all colimits are generated by finite and filtered ones. □