A1. On monoidal categories

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1 On monoidal categories

1.1 R-linear monoidal categories

Definition 1.1. Let R be a ring. An *R-linear monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \phi)$ (Stacks project authors, 2020, Tag 0FFJ) such that $\mathcal{C}$ is R-linear (Stacks project authors, 2020, Tag 09MI) and the functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is R-bilinear, i. e., for any objects X, Y, Z, W of $\mathcal{C}$ the map

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \times \operatorname{Hom}_{\mathcal{C}}(Z, W) \to \operatorname{Hom}_{\mathcal{C}}(X \otimes Z, Y \otimes W)$$

is R-bilinear.

Definition 1.2. A *functor of R-linear monoidal categories* $F\colon \mathcal{C} \to \mathcal{C}'$ is given by a functor of monoidal categories (Stacks project authors, 2020, Tag 0FFL) that is also R-linear (Stacks project authors, 2020, Tag 09MK). I. e., there is a natural transformation $\otimes' \circ (F \times F) \simeq F \circ \otimes$ satisfying an associativity condition, $F(\mathbf{1})$ is a unit in $\mathcal{C}'$, and $\operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is an R-module homomorphism for all objects X and Y of $\mathcal{C}$.

Definition 1.3. An *R-linear symmetric monoidal category* is a quadruple $(\mathcal{C}, \otimes, \phi, \psi)$, where $(\mathcal{C}, \otimes, \phi)$ is an R-linear monoidal category and ψ is a commutativity constraint compatible with ϕ. I. e., $(\mathcal{C}, \otimes, \phi, \psi)$ is a symmetric monoidal category that is R-linear.

Definition 1.4. A *functor of R-linear symmetric monoidal categories* $F\colon \mathcal{C} \to \mathcal{C}'$ is a functor of symmetric monoidal categories (Stacks project authors, 2020, Tag 0FFY) that is R-linear.

Remark 1.5. Note that we do not need to require the associativity constraint ϕ and the commutativity constraint ψ to be R-linear as this will automatically be the case: More generally, suppose $F, G\colon \mathcal{A} \to \mathcal{B}$ are R-linear functors between R-linear categories $\mathcal{A}$ and $\mathcal{B}$. Then a natural transformation $\eta\colon F \to G$ from F to G is automatically R-linear in the following sense: For any object A of $\mathcal{A}$ and any $r \in R$ the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "r\cdot\eta_A"] \arrow[d, "r\cdot\mathrm{id}_{F(A)}"'] & G(A) \arrow[d, "\mathrm{id}_{G(A)}"] \\
F(A) \arrow[r, "\eta_A"'] & G(A)
\end{tikzcd}$$

The top horizontal arrow uses the R-module structure of $\operatorname{Hom}_{\mathcal{B}}(F(A), G(A))$ and the left vertical arrow that of $\operatorname{Hom}_{\mathcal{B}}(F(A), F(A))$.

1.2 Rigidity

Definition 1.6. Given an R-linear symmetric monoidal cateogry $(\mathcal{C}, \otimes, \phi, \psi)$ and an object X of $\mathcal{C}$, a *dual* of X is a left or right dual of X in $\mathcal{C}$ viewed as a monoidal category (Stacks project authors, 2020, Tag 0FFP), i e., an object X' of $\mathcal{C}$ together with morphisms $\eta\colon \mathbf{1} \to X \otimes X'$ (*unit/coevaluation map*) and $\epsilon\colon X' \otimes X \to \mathbf{1}$ (*counit/evaluation map*) such that

$$\begin{tikzcd} X \arrow[r, "\eta\otimes \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"'] & X \otimes X' \otimes X \arrow[d, "\mathrm{id}_X \otimes \epsilon"] \\ & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "\mathrm{id}_{X'} \otimes \epsilon"] \arrow[dr, "\mathrm{id}_{X'}"'] & X \otimes X' \otimes X \arrow[d, "\epsilon\otimes \mathrm{id}_{X'}"] \\ & Y \end{tikzcd}$$

commute (for a left dual X') or with X and X' interchanged for a right dual. By (Stacks project authors, 2020, Tag 0FN8) any left dual X' of X will also be a right dual, so in the case of symmetric monoidal categories we simply refer to *duals* and omit right/left. If a dual of X exists, we denote if by $X^\vee$ and call X *dualizable*.

Remark 1.7. Suppose the object X of $\mathcal{C}$ has a dual $X^\vee$, then by (Stacks project authors, 2020, Tag 0FFQ) the functor $- \otimes X$ has right and left adjoint $- \otimes X^\vee$. In particular there is a natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(Z \otimes X, Y) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y \otimes X^\vee)$$

compatible with $\otimes$ and functorial in both Z and Y. This motivates the following definition.

Definition 1.8. Suppose the object X of the R-linear symmetric monoidal cateogry $\mathcal{C}$ has a dual $X^\vee$ in the sense of 1.6, then the *inner hom* of X and $Y \in \mathrm{ob}(\mathcal{C})$ is $\underline{\mathrm{Hom}}(X, Y) \coloneqq Y \otimes X^\vee$. By the previous remark, this yields a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Z \otimes X, Y) \to \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(X, Y))$$

for any $Z \in \mathrm{ob}(\mathcal{C})$.

Definition 1.9. A symmetric monoidal category $\mathcal{C}$ is called *rigid* if all objects X of $\mathcal{C}$ are *dualizable*.

Definition 1.10. For a finite dualizable object X of $(\mathcal{C}, \otimes, \phi, \psi)$ we call the composition

$$\mathrm{End}(X) = \mathrm{Hom}(\mathbf{1}, X \otimes X^\vee) \xrightarrow{\mathrm{Hom}(1,\psi)} \mathrm{Hom}(\mathbf{1}, X^\vee \otimes X) \to \mathrm{End}(\mathbf{1})$$

the *trace morphism of* X and denote it by tr_X. The *dimension* of X is $\mathrm{tr}_X(\mathrm{id}_X) \in \mathrm{End}(\mathbf{1})$.

1.3 Abelian symmetric monoidal categories

Definition 1.11. An *abelian* symmetric monoidal categoy is a symmetric monoidal category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is an abelian category and the functor $\otimes$ is additive in each variable.

Proposition 1.12. *Let $(\mathcal{C}, \otimes)$ be a rigid symmetric monodial category such that $\mathcal{C}$ is abelian. Then the functor $\otimes$ is bi-additive commutes with colimits and limits in each variable.*

Proof. By assumption any object X of $\mathcal{C}$ is dualizable. The functor $- \otimes X\colon \mathcal{C} \otimes \mathcal{C}$ is left and right adjoint to $- \otimes X^\vee$, hence it commutes with colimits and limits, and it is additive. $\square$

1.4 Ind-Completion

Definition 1.13. For a category $\mathcal{C}$ let $\mathrm{PSh}(\mathcal{C})$ denote the category of presheaves of sets on $\mathcal{C}$ (Stacks project authors, 2020, Tag 00V1). An *ind-object* in $\mathcal{C}$ is an object of $\mathrm{PSh}(\mathcal{C})$ which is isomorphic to a filtered colimit (Stacks project authors, 2020, Tag 04AX) $\mathrm{colim}_I\, h_{\mathcal{C}}(M)$ for $M\colon I \to \mathcal{C}$ a filtered diagram in $\mathcal{C}$ and $h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ the Yoneda embedding.

Definition 1.14. The *ind-completion of* $\mathcal{C}$ is the full subcategory $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ on ind-objects in $\mathcal{C}$. Denote by $i_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ the natural functor induced by $h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$.

Proposition 1.15. *The category* $\mathrm{Ind}(\mathcal{C})$ *admits all small filtered colimits and the inclusion* $\mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ *commutes with small filtered colimits.*

Proposition 1.16. *The ind-completion of an R-linear symmetric monoidal category $\mathcal{C}$ acquires a canonical symmetric monoidal structure extending that of $\mathcal{C}$.*

Proof. Use Proposition 1.12. □

References

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