A1. On k-linear categories

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1 On monoidal categories

1.1 R-linear monoidal categories

Definition 1.1. Let R be a ring. An *R-linear monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \phi)$ (Stacks project authors, 2020, Tag 0FFJ) such that $\mathcal{C}$ is R-linear (Stacks project authors, 2020, Tag 09MI) and the functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is R-bilinear, i. e., for any objects X, Y, Z, W of $\mathcal{C}$ the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Z, W) \to \mathrm{Hom}_{\mathcal{C}}(X \otimes Z, Y \otimes W)$$

is R-bilinear.

Definition 1.2. An *R-linear symmetric monoidal category* is a quadruple $(\mathcal{C}, \otimes, \phi, \psi)$, where $(\mathcal{C}, \otimes, \phi)$ is an R-linear monoidal category and ψ is a commutativity constraint compatible with ϕ. I. e., $(\mathcal{C}, \otimes, \phi, \psi)$ is a symmetric monoidal category that is R-linear.

1.2 Internal Hom

Definition 1.3. Let X and Y be objects of a monoidal category $\mathcal{C}$. If the functor

$$\mathrm{Hom}_{\mathcal{C}}(- \otimes X, Y)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets},\ Z \mapsto \mathrm{Hom}_{\mathcal{C}}(Z \otimes X, Y)$$

is representable, we denote the representing object by $\underline{\mathrm{Hom}}(X, Y) \in \mathrm{ob}(\mathcal{C})$ and call it the *inner hom* from X to Y. This means that there is an isomorphism, functorial in Z:

$$\mathrm{Hom}_{\mathcal{C}}(Z \otimes X, Y) \to \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(X, Y)).$$

The unique preimage of $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$ under the functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}(X, Y) \otimes X, Y) \to \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y))$$

is denoted by

$$\mathrm{ev}_{X,Y}\colon\ \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y.$$

Remark 1.4.

$$\mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(X, Y)) \cong \mathrm{Hom}(\mathbf{1} \otimes X, Y) = \mathrm{Hom}(X, Y).$$

Definition 1.5. If for an object X of $\mathcal{C}$ and the unit object $\mathbf{1}$ the functor $\mathrm{Hom}_{\mathcal{C}}(X \otimes -, \mathbf{1})$ is representable, we denote the inner hom $\underline{\mathrm{Hom}}(X, \mathbf{1})$ by $X^\vee$ and call it the *dual object* to X. We call X *dualizable.* We denote $\mathrm{ev}_X \coloneqq \mathrm{ev}_{X,\mathbf{1}} \colon X^\vee \otimes X \to \mathbf{1}$ and call it the *evaluation morphism.* For any object Z of $\mathcal{C}$ we have a functorial isomorphism

$$\mathrm{Hom}(Z, X^\vee) \to \mathrm{Hom}(Z \otimes X, \mathbf{1}).$$

If objects X and Y of $\mathcal{C}$ are dualizable and $f \colon X \to Y$ is a morphism in $\mathcal{C}$ we can define ${}^t f \colon Y^\vee \to X^\vee$ as the image of $\mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f)$ under the functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Y^\vee \otimes X, \mathbf{1}) \to \mathrm{Hom}_{\mathcal{C}}(Y^\vee, X^\vee).$$

When f is an isomorphism, so is ${}^t f$ and we let $f^\vee \coloneqq ({}^t f)^{-1} \colon X^\vee \to Y^\vee$. Then

$$\mathrm{ev}_Y \circ (f^\vee \otimes f) = \mathrm{ev}_X \colon X^\vee \otimes X \to \mathbf{1}.$$

Definition 1.6. In a symmetric monoidal category $\mathcal{C}$, let $i_X \colon X \to X^{\vee\vee}$ be the unique preimage of the composition of the commutativity law $\psi \colon X \otimes X^\vee \to X^\vee \otimes X$ with $\mathrm{ev}_X \colon X^\vee \otimes X \to \mathbf{1}$ under the functorial isomorphism

$$\mathrm{Hom}(X, X^{\vee\vee}) \to \mathrm{Hom}(X \otimes X^\vee, \mathbf{1}) \ni \mathrm{ev}_X \circ \psi.$$

If i_X is an isomorphism, then X is called *reflexive.*

Definition 1.7. A symmetric monoidal category $\mathcal{C}$ is called *rigid* if

1. the inner hom $\underline{\mathrm{Hom}}(X, Y)$ exists for all objects X and Y,
2. the morphisms

   $$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \to \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

   corresponding to the morphism

   $$\big(\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2)\big) \otimes \big(X_1 \otimes X_2\big) \xrightarrow{\mathrm{ev}_{X_1,Y_1} \otimes \mathrm{ev}_{X_2,Y_2}} Y_1 \otimes Y_2$$

   are isomorphisms for all objects X_1, X_2, Y_1, Y_2 of $\mathcal{C}$,
3. all objects of $\mathcal{C}$ are reflexive.

Remark 1.8. A symmetric monoidal category $\mathcal{C}$ is *rigid* if and only if all objects in $\mathcal{C}$ admit a dual: If all objects admit a dual, then the inner hom for objects X and Y of $\mathcal{C}$ is the objects $X^\vee \otimes Y$ with $\mathrm{ev}_{X,Y} \colon (X^\vee \otimes Y) \otimes X \xrightarrow{\psi} X^\vee \otimes X \otimes Y \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_Y} \mathbf{1} \otimes Y \to Y$.

Definition 1.9. Let X and Y be objects of a symmetric monoidal category $\mathcal{C}$ such that the inner hom $\underline{\mathrm{Hom}}(X, Y)$ exists and X is dualizable. The morphism

$$(X^\vee \otimes Y) \otimes X \cong \big(\underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes \underline{\mathrm{Hom}}(\mathbf{1}, Y)\big) \otimes (X \otimes \mathbf{1}) \to \mathbf{1} \otimes Y \cong Y$$

corresponds to a morphism $\phi_{X,Y}$:

$$X^\vee \otimes Y = \underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes \underline{\mathrm{Hom}}(1, Y) \to \underline{\mathrm{Hom}}(X \otimes \mathbf{1}, \mathbf{1} \otimes Y) = \underline{\mathrm{Hom}}(X, Y).$$

An object X of $\mathcal{C}$ is called *finite* if the morphism $\phi_{X,X} \colon X^\vee \otimes X \to \underline{\mathrm{Hom}}(X, X)$ is an isomorphism.

Definition 1.10. For a finite dualizable object X of $\mathcal{C}$ we call the composition

$$\underline{\mathrm{Hom}}(X,X) \xrightarrow{\phi_{X,X}^{-1}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}.$$

the *trace morphism of* X and denote it by tr_X. The *dimension* of X is the composition of the trace tr_X with $j_X\colon \mathbf{1} \to \underline{\mathrm{Hom}}(X,X)$ (induced by $X \otimes 1 \to X$):

$$\mathrm{End}(\mathbf{1},\mathbf{1}) \ni \dim_X\colon \mathbf{1} \xrightarrow{j_X} \underline{\mathrm{Hom}}(X,X) \xrightarrow{\mathrm{tr}_X} \mathbf{1}$$

[1]

1.3 Abelian symmetric monoidal categories

Definition 1.11. An *abelian* symmetric monoidal categoy is a symmetric monoidal category $(\mathcal{C},\otimes)$ such that $\mathcal{C}$ is an abelian category and the functor $\otimes$ is additive in each variable.

Proposition 1.12. *Let $(\mathcal{C},\otimes)$ be a rigid symmetric monodial category such that $\mathcal{C}$ is abelian. Then the functor $\otimes$ is bi-additive commutes with colimits and limits in each variable.*

Proof. The functor $-\otimes X\colon \mathcal{C}\otimes\mathcal{C}$ is left adjoint to $\underline{\mathrm{Hom}}(X,-)$, hence it commutes with colimits and is additive. By considering the opposite caegory $\mathcal{C}^{\mathrm{op}}$ we see that $-\otimes X$ is also right adjoint to $\underline{\mathrm{Hom}}(X,-)$, so it also preserves limits. □

1.4 Ind-Completion

Definition 1.13. For a category $\mathcal{C}$ let $\mathrm{PSh}(\mathcal{C})$ denote the category of presheaves of sets on $\mathcal{C}$ (Stacks project authors, 2020, Tag 00V1). An *ind-object* in $\mathcal{C}$ is an object of $\mathrm{PSh}(\mathcal{C})$ which is isomorphic to a filtered colimit (Stacks project authors, 2020, Tag 04AX) $\mathrm{colim}_I\, h_{\mathcal{C}}(M)$ for $M\colon I \to \mathcal{C}$ a filtered diagram in $\mathcal{C}$ and $h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ the Yoneda embedding.

Definition 1.14. The *ind-complection of* $\mathcal{C}$ is the full subcategory $\mathrm{Ind}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ on ind-objects in $\mathcal{C}$. Denote by $i_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ the natural functor induced by $h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$.

Proposition 1.15. *The category* $\mathrm{Ind}(\mathcal{C})$ *admits all small filtered colimits and the inclusion* $\mathrm{Ind}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ *commutes with small filtered colimits.*

Proposition 1.16. *The ind-completion of an R-linear symmetric monoidal category $\mathcal{C}$ acquires a canonical symmetric monoidal structure extending that of $\mathcal{C}$.*

Proof. Use Proposition 1.12. □

Definition 1.17. faithfully flat object of an R-linear monoidal abelian category

[1] Or should we define the trace as $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},-)$ applied to tr_X? Note that $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\underline{\mathrm{Hom}}(X,Y)) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Then it is a morphism $\mathrm{Tr}_X\colon \mathrm{End}(X) \to \mathrm{End}(\mathbf{1})$ and $\dim(X) = \mathrm{Tr}_X(\mathrm{id}_X)$. Compare (Deligne and Milne, 1982, p. 10).

References

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