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On Tannakian duality over valuation rings

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Abstract

In this paper we generalize the Tannakian theory which gives a correspondence between groupoids and Tannakian categories over a field k to the case where k is a valuation ring. We give a general theorem how to reconstruct groupoids in arbitrary categories from their category of representations and we show that this theorem can be applied to groupoids over Dedekind rings. We also give a partial answer how to see whether a category is the representation category of a groupoid over a valuation ring.

Introduction

In 1939, Tannaka established a duality between compact groups and their representations [15]. He proved that a compact group is already determined by its unitary dual. In 1972, Saavedra, using ideas of Grothendieck, developed a "Tannakian theory" by establishing a functorial correspondence between gerbes over arbitrary fields which are tied by an affine group scheme and their representation category [11]. In particular, he obtained a duality between affine group schemes over a field and so called neutral Tannakian categories. A gap in Saavedra's proof was closed by Deligne in 1990 [3]. This way we get a correspondence between properties of affine group schemes (or certain gerbes) and proper-

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ties of their representation categories. Part of this dictionary has been developed by Deligne and Milne [4].

One can divide the duality established by Saavedra and Deligne in two parts:

(1) The reconstruction problem: given a "group-like" object G (e.g., a group scheme, a gerbe, a quantum group), is it possible to recover G from the category of its representations, using the forgetful functor?
(2) The description problem: give a purely categorical description which ensures that a given category is equivalent to a category of representations of some "group-like" object.

The description problem has been solved in a satisfying manner only for gerbes tied by affine group schemes over fields of characteristic zero [3, 7]. The reconstruction problem is much better understood. It has been generalized to quantum groupoids and braided groups (e.g., by Majid [9] and Bruguières [2]) over fields. Majid has also given a general categorical approach [7] but unfortunately his hypotheses are very restrictive.

Further, all those "group-like" objects were required to be over fields. But in mathematics there are also lots of such objects over more general rings (e.g., over the p-adic completion of $\mathbb{Z}$ or over $\mathbb{C}[\![q]\!]$).

The goal of this work is therefore three-fold. First, to give a general categorical method to reconstruct "group-like" objects from their category of representations even in the non-neutral case. This is obtained by 2.14 and 2.18. The second goal is to use this general and purely formal theorem to recover affine groupoids (and in particular affine group schemes) over Dedekind rings (or more generally Prüfer rings) from their category of representations (5.13 and 5.17). The last objective is to give a partial answer about the description problem for groupoids over discrete valuation rings (or more generally over valuation rings of height at most one) (6.18 and 6.20).

The reason why we work in the maybe somewhat unfamiliar setting of Prüfer rings and valuation rings of height one (instead of their noetherian counterparts, Dedekind rings and discrete valuation rings) is the following. First of all they occur in mathematical applications (e.g., the integral closure of a discrete valuation ring in an algebraic closure of its field of fractions). Further, if R is a valuation ring of height one, a fibre functor of an R-linear representation category exists in general only over some non-noetherian valuation ring of height one, even if R itself is noetherian.

I will now give an overview of the structure of this work. In the first section some categorical notions are recalled. Section 2 considers the main tool for the solution of the reconstruction problem, the comonoid of coendomorphisms of a functor. This notion goes back to MacLane [10]. The definition and the statement of the basic properties here are obvious generalizations of [3, §4]. The abstract reconstruction theorem (2.14) is therefore a formal generalization of [3, 4.13]. It describes how to recover a comonoid from its representation category using the forgetful functor. In case we started with a Hopf monoid (e.g., the Hopf algebra associated to some group scheme) we also recover the Hopf structure (2.18). Here we refer to [7] for the necessary diagram chasing.

The next two sections collect some tools to attack the reconstruction of groupoids over Prüfer rings. In the third section we give some basic definitions and properties of R-linear monoidal categories and the notion skalar extension. In the fourth section we describe the connection of groupoids and gerbes over arbitrary schemes. Most of these properties are easy generalizations of [3] where the case of groupoids and gerbes over a field is considered.

The fifth section starts with the description of coalgebroids and their comodules. We check that all conditions of the abstract reconstruction theorem (2.14) are satisfied over a Prüfer ring. For this we have to show some properties of comodules over Prüfer rings. Most of these properties are well known for Dedekind rings (e.g., [13]), and most of the time the proofs are easy modifications. After these technical lemmas we obtain the reconstruction theorem for coalgebroids (5.13) and for affine groupoids (and hence for affine group schemes) (5.17).

In the last section we define the notion of a Tannakian lattice over a valuation ring of height at most one (6.9). Roughly speaking it is a rigid pseudo-abelian symmetric monoidal category which admits a fibre functor ω over some faithfully flat R-scheme such that the skalar extension of the category to the field of fraction of R is a Tannakian category in the sense of [3]. We show that the category of representations of a groupoid is in fact a Tannakian lattice (6.17). For this we use the theory of gerbes provided by Section 5 and the fact that every Tannakian lattice has a fibre functor over a sufficiently "nice" R-scheme (6.14). The main theorem of this section is 6.18 which assures that the fibre functor always provides a fully faithful embedding of a Tannakian lattice into the category of representation of a groupoid G and that G is universal with this property. We conclude with a corollary for the neutral case (6.20).

1. Monoidal categories

1.1. By a *monoidal category* we mean a tuple $(\mathcal{M}, \otimes, 1, \alpha, \lambda, \rho)$ where $\mathcal{M}$ is a category, $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a bifunctor, $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ is an associativity constraint, 1 is a unit object, $\lambda_X : X \otimes 1 \xrightarrow{\sim} X$ is a left unit constraint, and where $\rho_X : X \otimes 1 \xrightarrow{\sim} X$ is right unit constraint. These are to satisfy

(a) (Pentagon axiom): $(\mathrm{id}_X \otimes \alpha_{Y,Z,W}) \circ \alpha_{X,Y\otimes Z,W} \circ (\alpha_{X,Y,Z} \otimes \mathrm{id}_W) = \alpha_{X,Y,Z\otimes W} \circ \alpha_{X\otimes Y,Z,W}$.
(b) (Unit axiom): $(\mathrm{id}_X \otimes \lambda_Y) \circ \alpha_{X,1,Y} = \rho_X \otimes \mathrm{id}_Y$.

By abuse of notation we will often simply write $\mathcal{M}$ for the monoidal category. A monoidal category is called *symmetric*, if there is given a commutativity constraint whose square is the identity. We also have the weaker notion of a *braided* monoidal category. We refer to [8] for the precise definition.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two monoidal categories. A *functor* $\mathcal{M}_1 \to \mathcal{M}_2$ *of monoidal categories* (or a *monoidal functor*) is a functor $T : \mathcal{M}_1 \to \mathcal{M}_2$ together with a functorial isomorphism $T(X) \otimes T(Y) \xrightarrow{\sim} T(X \otimes Y)$ which is compatible with the associativity constraint equipped with an isomorphism $T(1_{\mathcal{M}_1}) \xrightarrow{\sim} 1_{\mathcal{M}_2}$ compatible with the unit constraints. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are braided (or symmetric) we call a monoidal functor a *tensor functor* if it is compatible with the braiding.

From now on we denote by $\mathcal{M}$ a monoidal category. Moreover, we assume that all monoidal categories are *strict*, i.e., all constraints are the identity. By the coherence theorem (e.g., [10, Chapter XI]) every monoidal category is equivalent to a strict monoidal category. If $\mathcal{M}$ is a symmetric monoidal category we also can and will assume by loc. cit. that the commutativity constraint is the identity. Similar for braided monoidal categories.

1.2. Let X be an object of $\mathcal{M}$. A (*left*) *dual object of* X is a triple $(X^\vee, \mathrm{ev}, \delta)$ where $X^\vee$ is an object of $\mathcal{M}$ and where $\mathrm{ev}: X \otimes X^\vee \to 1$ and $\delta: 1 \to X^\vee \otimes X$ are morphisms such that

$$X \xrightarrow{\mathrm{id}_X \otimes \delta} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_X} X,$$
$$X^\vee \xrightarrow{\delta \otimes \mathrm{id}_{X^\vee}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \otimes \mathrm{ev}} X^\vee$$

are the identity.

Note that δ is uniquely determined. Further, a dual $(X^\vee, \mathrm{ev}, \delta)$ is unique up to unique isomorphism.

We call an object X *rigid* if there exists a dual of X. We call $\mathcal{M}$ *rigid* if every object in $\mathcal{M}$ is rigid.

1.3. Let X and Y be objects in $\mathcal{M}$. If X and Y admit dual objects $X^\vee$ and $Y^\vee$, then $Y^\vee \otimes X^\vee$ is a dual of $X \otimes Y$ where $\mathrm{ev}_{X\otimes Y}$ is given by $\mathrm{ev}_X \circ (\mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^\vee})$. In this case $\delta_{X\otimes Y}$ is given by $(\mathrm{id}_X \otimes \delta_Y \otimes \mathrm{id}_{X^\vee}) \circ \delta_X$.

In particular, the full subcategory of rigid objects of $\mathcal{M}$ inherits the structure of a monoidal category.

1.4. Example. Let A be a commutative ring. Then the category of A-modules endowed with the usual tensor structure is a symmetric monoidal category. An A-module M is rigid if and only if M is finitely generated projective. In this case we have $M^\vee = \mathrm{Hom}(M, A)$ and (using the identification $M^\vee \otimes M = \underline{\mathrm{End}}(M)$) ev_M (respectively δ_M) are given by the usual evaluation (respectively by the map $A \to \mathrm{End}(M)$, $a \mapsto a\,\mathrm{id}_M$).

1.5. Let $T: \mathcal{M}_1 \to \mathcal{M}_2$ be a functor of monoidal categories and let X be an object which admits a dual $(X^\vee, \mathrm{ev}, \delta)$. Then $T(X)$ admits a dual which is given by $(T(X^\vee), T(\mathrm{ev}), T(\delta))$. In particular, T induces a functor from the full subcategory of rigid objects in $\mathcal{M}_1$ into the full subcategory of rigid objects of $\mathcal{M}_2$.

1.6. A *comonoid* in $\mathcal{M}$ is a triple (X, c, e) consisting of an object X of $\mathcal{M}$, a comultiplication $c: X \to X \otimes X$, and a morphism $e: X \to 1$ such that

(COM1) c is coassociative, i.e., the compositions

$$X \xrightarrow{c} X \otimes X \overset{\mathrm{id}\otimes c}{\underset{c\otimes \mathrm{id}}{\rightrightarrows}} X \otimes X \otimes X$$

are equal.

(COM2) e is a counit, i.e., the compositions

$$X \xrightarrow{c} X \otimes X \underset{e\otimes\mathrm{id}}{\overset{\mathrm{id}\otimes e}{\rightrightarrows}} X$$

are the identity.

Note that the counit is unique if it exists. We have the dual notion of a *monoid* in a monoidal category.

Let $\mathcal{M}$ be a braided monoidal category. A *bimonoid* in $\mathcal{M}$ is an object X which carries the structure of a monoid (X, m, u) and of a comonoid (X, c, e) such that c and e are morphisms of monoids.

Finally a *Hopf monoid* in $\mathcal{M}$ is bimonoid (X, m, u, c, e) together with an antipode $\iota : X \to X$ satisfying the usual conditions for a Hopf algebra (see, e.g., [14]).

1.7. Let $\mathcal{M}$ be a braided monoidal category and let (X, c, e) and (X', c', e') be two comonoids in $\mathcal{M}$. Then the compositions

$$X \otimes X' \xrightarrow{c\otimes c'} (X \otimes X) \otimes \big(X' \otimes X'\big) = \big(X \otimes X'\big) \otimes \big(X \otimes X'\big),$$
$$X \otimes X' \xrightarrow{e\otimes e'} 1 \otimes 1 = 1$$

define the structure of a comonoid on $X \otimes X'$. This is trivial if $\mathcal{M}$ is symmetric. For the general case we refer to [8, 2.1].

1.8. Definition. A *category with right $\mathcal{M}$-action* is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$ and functorial compatible isomorphisms $X \otimes (M \otimes N) \xrightarrow{\sim} (X \otimes M) \otimes N$ and $X \otimes 1 \xrightarrow{\sim} X$ for M, N objects of $\mathcal{M}$ and X object of $\mathcal{C}$. Again we can and will assume by the coherence theorem that these functorial isomorphisms are the identity.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories with $\mathcal{M}$-action. A functor $\omega : \mathcal{C} \to \mathcal{C}'$ together with a functorial isomorphism $\xi : \omega(X \otimes M) \to \omega(X) \otimes M$ for $M \in \mathcal{M}$ and $X \in \mathcal{C}$ is called an *$\mathcal{M}$-functor*.

More generally, let $h : \mathcal{M} \to \mathcal{M}'$ be a functor of monoidal categories and let $\mathcal{C}$ (respectively $\mathcal{C}'$) be a category with right $\mathcal{M}$-action (respectively right $\mathcal{M}'$-action). A functor $\omega : \mathcal{C} \to \mathcal{C}'$ together with a functorial isomorphism $\xi : \omega(X \otimes M) \to \omega(X) \otimes h(M)$ is called an *h-functor*.

Let (ω, ξ) and (ω', ξ') be two $\mathcal{M}$-functors. A morphism of functors $\varphi : \omega \to \omega'$ is called a *morphism of $\mathcal{M}$-functors* or shorter an *$\mathcal{M}$-morphism* if φ commutes with ξ and ξ'. The set of $\mathcal{M}$-morphisms $\omega \to \omega'$ is denoted by $\mathrm{Hom}_{\mathcal{M}}(\omega, \omega')$.

This way we get the 2-category of categories with $\mathcal{M}$-action.

1.9. Definition. Let $\mathcal{M}$ be a monoidal category and let $\mathcal{C}$ be a category with a right $\mathcal{M}$-action. Let (L, c, e) be a comonoid in $\mathcal{M}$. A pair (X, r) consisting of an object X in $\mathcal{C}$ and a morphism $r : X \to X \otimes L$ is called *L-comodule* if it satisfies the following conditions:

(CM1) r is compatible with the coproduct, i.e., the compositions

$$X \xrightarrow{r} X \otimes L \underset{r\otimes \mathrm{id}_L}{\overset{\mathrm{id}_X\otimes c}{\rightrightarrows}} X \otimes L \otimes L$$

are equal.

(CM2) r is compatible with the counit, i.e., $(\mathrm{id}_X \otimes e) \circ r = \mathrm{id}_X$.

A *homomorphism of L-comodules* $(X, r) \to (X', r')$ is a morphism $\varphi : X \to X'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "r"'] & X' \arrow[d, "r'"] \\ X \otimes L \arrow[r, "\varphi\otimes \mathrm{id}_L"] & X' \otimes L. \end{tikzcd}$$

We denote the category of L-comodules in $\mathcal{C}$ by $\mathcal{C}^L$.

1.10. A monoidal category acts on itself. More generally, let C be a comonoid in $\mathcal{M}$. Then the monoidal structure of $\mathcal{M}$ induces a left $\mathcal{M}$-action on the category $\mathcal{M}^C$ of right C-comodules.

If $f : C \to C'$ is a homomorphism of comonoids in $\mathcal{M}$, the induced functor $\mathcal{M}^C \to \mathcal{M}^{C'}$ is an $\mathcal{M}$-functor.

1.11. We keep the notations of 1.9. Let X be any object of $\mathcal{C}$. Then

$$X \otimes L \xrightarrow{\mathrm{id}_X\otimes c} X \otimes L \otimes L$$

defines the structure of an L-comodule on the object $X \otimes L$ of $\mathcal{C}$. This way

$$X \mapsto X \otimes L, \qquad f \mapsto f \otimes \mathrm{id}_L$$

defines a functor $\mathcal{C} \to \mathcal{C}^L$. This functor is right adjoint to the forgetful functor $\mathcal{C}^L \to \mathcal{C}$. Indeed, for every object X of $\mathcal{C}$ and every L-comodule (Y, r_Y) the maps

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}^L}(Y, X \otimes L), \quad f \mapsto (f \otimes \mathrm{id}_L) \circ r_Y,$$
$$\mathrm{Hom}_{\mathcal{C}^L}(Y, X \otimes L) \to \mathrm{Hom}_{\mathcal{C}}(Y, X), \quad g \mapsto (\mathrm{id}_X \otimes e) \circ g,$$

are functorial and inverse to each other.

Now assume that X itself is an L-comodule and denote by r its coaction. Then $r : X \to X \otimes L$ is a homomorphism of comodules. Further we have $(e \otimes 1) \circ r = \mathrm{id}_X$, i.e., r is a section of $e \otimes 1$.

2. The comonoid of coendomorphisms of functors

2.1. In this section we fix the following notations: let $\mathcal{M}$ be a monoidal category and $\mathcal{C}$ be a category with a right $\mathcal{M}$-action. We write $\widehat{\mathcal{M}}$ for the category of copresheaves on $\mathcal{M}$, i.e., the category of covariant functors from $\mathcal{M}$ in the category of sets. Denote by $\mathcal{D}$ an essentially small category and by $\omega_i : \mathcal{D} \to \mathcal{C}$ $(i = 1, 2, \ldots)$ a family of functors.

2.2. For every object M in $\mathcal{M}$ write $\omega_i \otimes M$ for the functor

$$\mathcal{D} \to \mathcal{C}, \quad X \mapsto \omega_i(X) \otimes M.$$

Then

$$M \mapsto \mathrm{Hom}(\omega_2, \omega_1 \otimes M)$$

is a copresheaf on $\mathcal{M}$. We denote it by $\underline{\mathrm{CoHom}}(\omega_1, \omega_2) = \underline{\mathrm{CoHom}}_{\mathcal{M}}(\omega_1, \omega_2)$. In this case we have a functorial isomorphism

$$\mathrm{Hom}(\omega_2, \omega_1 \otimes M) \cong \mathrm{Hom}_{\widehat{\mathcal{M}}}\big(\underline{\mathrm{CoHom}}(\omega_1, \omega_2), M\big).$$

In the case that the copresheaf is corepresentable, we denote the corepresenting object also by $\underline{\mathrm{CoHom}}(\omega_1, \omega_2) = \underline{\mathrm{CoHom}}_{\mathcal{M}}(\omega_1, \omega_2)$ and call it the *object of cohomomorphisms from ω_1 to ω_2*. Then we have the universal morphism of functors

$$\omega_2 \to \omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2). \tag{2.2.1}$$

Finally, we set

$$\underline{\mathrm{CoEnd}}_{\mathcal{M}}(\omega_i) := \underline{\mathrm{CoEnd}}(\omega_i) := \underline{\mathrm{CoHom}}(\omega_i, \omega_i).$$

2.3. From now on, we assume that all $\underline{\mathrm{CoHom}}(\omega_i, \omega_j)$ are corepresentable for $i \leqslant j$. Iterating (2.2.1), we get a morphism of functors

$$\omega_3 \to \omega_2 \otimes \underline{\mathrm{CoHom}}(\omega_2, \omega_3) \to \omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoHom}}(\omega_2, \omega_3)$$

and therefore a morphism

$$\underline{\mathrm{CoHom}}(\omega_1, \omega_3) \to \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoHom}}(\omega_2, \omega_3). \tag{2.3.1}$$

This "coproduct" is coassociative, i.e., we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{CoHom}}(\omega_1, \omega_4) \arrow[r] \arrow[d] & \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoHom}}(\omega_2, \omega_4) \arrow[d] \\
\underline{\mathrm{CoHom}}(\omega_1, \omega_3) \otimes \underline{\mathrm{CoHom}}(\omega_3, \omega_4) \arrow[r] & \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoHom}}(\omega_2, \omega_3) \otimes \underline{\mathrm{CoHom}}(\omega_3, \omega_4).
\end{tikzcd}$$

Finally, $\mathrm{id}_{\omega_i} \in \mathrm{Hom}(\omega_i, \omega_i) = \mathrm{Hom}(\omega_i, \omega_i \otimes 1)$ corresponds to a counit $\varepsilon_i : \underline{\mathrm{CoEnd}}(\omega_i) \to 1$, i.e., the compositions

$$\underline{\mathrm{CoHom}}(\omega_1, \omega_2) \to \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoEnd}}(\omega_2) \xrightarrow{\mathrm{id}\otimes\varepsilon_2} \underline{\mathrm{CoHom}}(\omega_1, \omega_2),$$
$$\underline{\mathrm{CoHom}}(\omega_1, \omega_2) \to \underline{\mathrm{CoEnd}}(\omega_1) \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \xrightarrow{\varepsilon_1\otimes\mathrm{id}} \underline{\mathrm{CoHom}}(\omega_1, \omega_2)$$

are the identity. Therefore, we see:

Proposition. *Let $\omega : \mathcal{D} \to \mathcal{C}$ be a functor, such that $\underline{\mathrm{CoEnd}}(\omega)$ is corepresentable. Then $\underline{\mathrm{CoEnd}}(\omega)$ is a comonoid in $\mathcal{M}$.*

2.4. Let $\omega : \mathcal{D} \to \mathcal{C}$ be a functor, such that $\underline{\mathrm{CoEnd}}(\omega)$ is corepresentable. Let C be any comonoid in $\mathcal{M}$. Then the bijection $\mathrm{Hom}(\underline{\mathrm{CoEnd}}(\omega), C) = \mathrm{Hom}(\omega, \omega \otimes C)$ induces an identification

$$\mathrm{Hom}_{CM}\big(\underline{\mathrm{CoEnd}}(\omega), C\big) = \mathrm{Hom}_{CM}(\omega, \omega \otimes C),$$

where the left-hand side denotes the set of comonoid homomorphism $\underline{\mathrm{CoEnd}}(\omega) \to C$ and where the right-hand side denotes the set of morphisms such that for every object X in $\mathcal{D}$ the induced arrow $\omega(X) \to \omega(X) \otimes C$ defines a C-comodule structure on $\omega(X)$.

2.5. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. For every object C' in $\mathcal{C}'$ we have the functor

$$\big(C' \otimes -\big) : \mathcal{M} \to \mathcal{C}, \quad M \mapsto C' \otimes M.$$

This induces a functor from $\mathcal{C}'$ into the category $\mathtt{Hom}(\mathcal{M}, \mathcal{C})$ of functors $\mathcal{M} \to \mathcal{C}$. We say that the action of $\mathcal{M}$ on $\mathcal{C}$ is *coclosed for $\mathcal{C}'$* if, for any $C' \in \mathcal{C}$, $(C' \otimes -)$ has a left adjoint functor. If this is the case, we denote by $F_{C'} : \mathcal{C} \to \mathcal{M}$ this left adjoint. Then $C' \mapsto F_{C'}$ defines a functor $\mathcal{C}' \to \mathtt{Hom}(\mathcal{C}, \mathcal{M})^{\mathrm{opp}}$.

2.6. Let $\mathcal{D}$ be the final category. Then to give ω is the same as to give an object X in $\mathcal{C}$. Assume that the functor $\mathcal{M} \to \mathcal{C}$ which sends M to $X \otimes M$ admits a left adjoint F_X (in other words, the action of $\mathcal{M}$ on $\mathcal{C}$ is coclosed for the subcategory which consists of X and id_X). Then $F_X(X)$ represents $\underline{\mathrm{CoEnd}}(\omega)$. Indeed, for every object M in $\mathcal{M}$ we have functorial bijections

$$\mathrm{Hom}\big(F_X(X), M\big) = \mathrm{Hom}(X, X \otimes M) = \mathrm{Hom}(\omega, \omega \otimes M) = \mathrm{Hom}\big(\underline{\mathrm{CoEnd}}(\omega), M\big).$$

In particular, $F_X(X)$ carries a comonoid structure. Further, by 2.4 we see that for every comonoid L in $\mathcal{M}$ morphisms of comonoids $F_X(X) \to L$ correspond to L-coactions on X.

2.7. Proposition. *Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ such that ω factorizes through $\mathcal{C}'$ and assume that the action of $\mathcal{M}$ on $\mathcal{C}$ is coclosed for $\mathcal{C}'$. Further suppose that there exist in $\mathcal{M}$ small inductive limits. Then $\underline{\mathrm{CoHom}}(\omega_1, \omega_2)$ is corepresentable.*

Proof. Denote by $F_{C'}$ the right adjoint of $(C' \otimes -)$ for $C' \in \mathrm{Ob}(\mathcal{C}')$. For every morphism $f: X \to Y$ in $\mathcal{D}$ define the category I_f as the subcategory of $\mathcal{M}$ consisting of three objects $F_{\omega_1(Y)}(\omega_2(X))$, $F_{\omega_1(X)}(\omega_2(X))$, and $F_{\omega_1(Y)}(\omega_2(Y))$ and the only morphisms in I_f (besides the identity morphisms) are

$$F_{\omega_1(f)}(\mathrm{id}): F_{\omega_1(Y)}\big(\omega_2(X)\big) \to F_{\omega_1(X)}\big(\omega_2(X)\big),$$
$$F_{\mathrm{id}}\big(\omega_2(f)\big): F_{\omega_1(Y)}\big(\omega_2(X)\big) \to F_{\omega_1(Y)}\big(\omega_2(Y)\big).$$

Denote by I the disjoint union of the categories I_f where f runs through all morphisms of $\mathcal{D}$. We have a canonical functor $I \to \mathcal{M}$, and it follows from 2.6 that its inductive limit represents $\underline{\mathrm{CoHom}}(\omega_1, \omega_2)$. □

2.8. Corollary. *Let $\mathcal{C} = \mathcal{M}$ be a braided monoidal category, and let $\omega_1, \omega_2 : \mathcal{D} \to \mathcal{M}$ be functors. Assume that ω_1 and ω_2 factorize through the full subcategory of rigid objects in $\mathcal{M}$ and that small inductive limits exist in $\mathcal{M}$. Then $\underline{\mathrm{CoHom}}(\omega_1, \omega_2)$ is corepresentable.*

Proof. For every rigid object C in $\mathcal{M}$ the functor $(C^\vee \otimes -)$ is left adjoint to $(C \otimes -)$ because we have $C = C^{\vee\vee}$ as $\mathcal{C}$ is symmetric. Therefore, the action of $\mathcal{M}$ on itself is coclosed for the subcategory of rigid objects in $\mathcal{M}$, and we can apply 2.7. □

2.9. Let $\mathcal{C}'$ be another category with a right action by a monoidal category $\mathcal{M}'$. Let $h: \mathcal{M} \to \mathcal{M}'$ be a tensor functor and let $f: \mathcal{C} \to \mathcal{C}'$ be an h-functor. Then the universal morphism $\omega_2 \to \omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2)$ induces by applying f a morphism

$$f \circ \omega_2 \to f \circ \big(\omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2)\big) = (f \circ \omega_1) \otimes h\big(\underline{\mathrm{CoHom}}(\omega_1, \omega_2)\big)$$

and therefore a canonical morphism of objects in $\mathcal{M}'$,

$$\underline{\mathrm{CoHom}}(f \circ \omega_1, f \circ \omega_2) \to h\big(\underline{\mathrm{CoHom}}(\omega_1, \omega_2)\big). \tag{2.9.1}$$

For $\omega_1 = \omega_2$ it follows by 2.4 that this is a morphism of comonoids in $\mathcal{M}'$.

2.10. We keep the notations of 2.9. Assume that h (respectively f) admits a right adjoint $h': \mathcal{M}' \to \mathcal{M}$ (respectively $f': \mathcal{C}' \to \mathcal{C}$) and that we are given an isomorphism of functors $\mathcal{C} \times \mathcal{M}' \to \mathcal{C}$,

$$\eta : \otimes \circ \big(\mathrm{id} \times h'\big) \cong f' \circ \otimes \circ f \times \mathrm{id} \tag{$*$}$$

such that the following diagram of functorial morphisms is commutative:

$$\begin{tikzcd}
(\mathrm{id} \times (h' \circ h)) \circ \otimes \arrow[r, "(\mathrm{id}\times h)\circ\eta"] & ((f' \circ f) \times \mathrm{id}) \circ \otimes \\
\otimes \arrow[u] \arrow[r, equal] & \otimes \arrow[u]
\end{tikzcd},$$

where the upper horizontal is derived from $(*)$ by composing both sides of $(*)$ with $\mathrm{id} \times h$ and where the vertical arrows are given by the adjunctions $\mathrm{id} \to h' \circ h$ and $\mathrm{id} \to f' \circ f$. Note that we assume neither that h' is a tensor functor nor that f' is an h'-functor.

Proposition. *With these notations and assumptions* (2.9.1) *is an isomorphism.*

Proof. Indeed, for every object M' in $\mathcal{M}'$ we have functorial isomorphisms

$$\begin{aligned}
\mathrm{Hom}\big(h\big(\underline{\mathrm{CoHom}}(\omega_1,\omega_2)\big), M'\big) &= \mathrm{Hom}\big(\underline{\mathrm{CoHom}}(\omega_1,\omega_2), h'\big(M'\big)\big)\\
&= \mathrm{Hom}\big(\omega_2, \omega_1 \otimes h'\big(M'\big)\big)\\
&\cong \mathrm{Hom}\big(\omega_2, f' \circ \big(\otimes M'\big) \circ f \circ \omega_1\big)\\
&= \mathrm{Hom}\big(f \circ \omega_2, (f \circ \omega_1) \otimes M'\big)\\
&= \mathrm{Hom}\big(\underline{\mathrm{CoHom}}(f \circ \omega_1, f \circ \omega_2), M'\big).
\end{aligned}$$

Setting $M' = h(\underline{\mathrm{CoHom}}(\omega_1,\omega_2))$, one sees that this functorial isomorphism gives an inverse of (2.9.1). $\square$

2.11. Let $\psi : \mathcal{D}' \to \mathcal{D}$ be a functor. The canonical morphism

$$\omega_2 \to \omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1,\omega_2)$$

defines for every object D' of $\mathcal{D}'$ a morphism $(\omega_2 \circ \psi)(D') \to (\omega_1 \circ \psi)(D') \otimes \underline{\mathrm{CoHom}}(\omega_1,\omega_2)$ and this gives

$$\underline{\mathrm{CoHom}}(\psi) : \underline{\mathrm{CoHom}}(\omega_1 \circ \psi, \omega_2 \circ \psi) \to \underline{\mathrm{CoHom}}(\omega_1,\omega_2).$$

For $\omega_1 = \omega_2$ this is a morphism of comonoids in $\mathcal{M}$ by 2.4.

2.12. Let $\mathcal{C} = \mathcal{M}$ be a tensor category, and let $\omega : \mathcal{D} \to \mathcal{M}$ be a functor. Assume that $C := \underline{\mathrm{CoEnd}}(\omega)$ is corepresentable. For every object X in $\mathcal{D}$ the image $\omega(X)$ carries a right C-comodule structure, i.e., ω factorizes through the category $\mathcal{M}^C$ of right C-modules in $\mathcal{M}$

$$\omega : \mathcal{D} \xrightarrow{\omega^C} \mathcal{M}^C \to \mathcal{M}.$$

Let L be any other comonoid in $\mathcal{M}$ such that ω factorizes through $\omega^L : \mathcal{D} \to \mathcal{M}^L$. This means that $\omega(X)$ is equipped with an L-comodule structure, functorial in X. Then by 2.4 there exists a unique homomorphism $C \to L$ of comonoids in $\mathcal{M}$ such that ω^L factorizes through the induced functor $\mathcal{M}^C \to \mathcal{M}^L$.

In particular, if $\mathcal{M}^L$ is equivalent to a small category, we can set $\mathcal{D} = \mathcal{M}^L$, and we see that the identity $\mathcal{M}^L \to \mathcal{M}^L$ factorizes in $\mathcal{M}^L \to \mathcal{M}^C \to \mathcal{M}^L$ where the second functor is given by the homomorphism $C \to L$ of comonoids.

2.13. Let $\mathcal{C}$ be a category with a right $\mathcal{M}$-action and let $\Phi : \mathcal{M} \to \mathcal{C}$ be a functor which is equipped with a functorial isomorphism $\Phi(X \otimes Y) \cong \Phi(X) \otimes Y$ which is compatible with the associativity and the units constraints. Let (L, c, e) be a comonoid in $\mathcal{M}$ and let $\mathcal{C}^L$ be the category of L-comodules in $\mathcal{C}$. The composition

$$\Phi(L) \xrightarrow{\Phi(c)} \Phi(L \otimes L) \cong \Phi(L) \otimes L$$

defines an L-comodule structure on $\Phi(L)$.

Now let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ and denote by $\mathcal{D}^L$ the category of L-comodules in $\mathcal{C}$ whose underlying object lies in $\mathcal{D}$. We have the canonical functor $\omega : \mathcal{D}^L \to \mathcal{D} \hookrightarrow \mathcal{C}$ of forgetting the coaction of L. We make the following assumptions:

(a) The category $\mathcal{D}^L$ is equivalent to a small category. The action of $\mathcal{M}$ on $\mathcal{C}$ is coclosed for $\mathcal{D}$ (2.5) and in $\mathcal{M}$ exist small inductive limits.
(b) Set $C = \underline{\mathrm{CoEnd}}(\omega)$. By (a) it is corepresentable by a comonoid in $\mathcal{C}$ (2.7). Assume that in $\mathcal{C}$ exist small filtered inductive limits and that the forgetful functors $\mathcal{C}^L \to \mathcal{C}$ and $\mathcal{C}^C \to \mathcal{C}$ reflect these (we already know that they preserve inductive limits because they admit a right adjoint (1.11)).
(c) The functors $X \mapsto X \otimes L$ and $X \mapsto X \otimes C$ from $\mathcal{C}$ into $\mathcal{C}$ commute with small filtered inductive limits.
(d) Every L-comodule in $\mathcal{C}^L$ is a small filtered inductive limit (in $\mathcal{C}^L$) of L-comodules in $\mathcal{D}^L$, and L itself is the filtered inductive limit of comonoids L_i in $\mathcal{M}$ such that the $\Phi(L_i)$ are in $\mathcal{D}$.
(e) The functor Φ is faithful and preserves and reflects filtered inductive limits.

2.14. Theorem (cf. [3, 4.13]). *We keep the assumptions of* 2.13. *By* 2.4 *we get a homomorphism of comonoids $u : C \to L$. This is an isomorphism.*

Proof. We first note that (b) implies that in $\mathcal{C}^L$ and $\mathcal{C}^C$ exist small filtered inductive limits. Let X be an object in $\mathcal{C}^L$, filtered inductive limit of L-comodules X_i in $\mathcal{D}^L$. By the universal property of C the functor ω factorizes through $\omega^C : \mathcal{D}^L \to \mathcal{C}^C$. Setting $\omega^C(X) := \varinjlim \omega^C(X_i)$, we get a functor

$$\omega^C : \mathcal{C}^L \to \mathcal{C}^C.$$

Note that ω^C commutes with the forgetful functors by (c).

By (d) we have $L = \varinjlim L_i$ for comonoids L_i in $\mathcal{M}$. By (e) we also have that $\Phi(L)$ is the inductive limit of the $\Phi(L_i)$. Applying ω^C we get a C-comodule structure $\Phi(L) \to \Phi(L) \otimes C$ on $\Phi(L)$. Because Φ reflects inductive limits and because of (c), this morphism is induced by a morphism $c' : L \to L \otimes C$ in $\mathcal{M}$. Define a as the composition

$$a : L \xrightarrow{c'} L \otimes C \xrightarrow{e \otimes \mathrm{id}_C} C.$$

We claim that a is an inverse of u. Using (c), one sees that $(\mathrm{id}_{\Phi(L)} \otimes u) \circ \Phi(c') = \Phi(c)$ (because this holds for $\Phi(L_i)$. Therefore, we have $\Phi(u) \circ \Phi(a) = \Phi(e \otimes \mathrm{id}_L) \circ \Phi(c) = \mathrm{id}_{\Phi(L)}$. As Φ is faithful this implies $u \circ a = \mathrm{id}_L$.

Now let (M, ρ) be an object of $\mathcal{D}^L$ and let $(M, \rho') = \omega^C((M, \rho))$. We can consider $\rho : M \to M \otimes L$ as homomorphism of L-comodules (1.11). Applying ω^C, ρ is also a homomorphism of C-comodules, i.e., the diagram

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho'"'] & M \otimes L \arrow[d, "\mathrm{id}_M \otimes c'"] \\ M \otimes C \arrow[r, "\rho \otimes \mathrm{id}_C"] & M \otimes L \otimes C \end{tikzcd}$$

is commutative. Composing $(\mathrm{id}_M \otimes c') \circ \rho$ and $(\rho \otimes \mathrm{id}_C) \circ \rho'$ from the left with $\mathrm{id}_M \otimes e \otimes \mathrm{id}_C$ we see that $\rho' = (\mathrm{id}_M \otimes a) \circ \rho$. The morphism $F_M(M) \to C$ corresponding to ρ' (2.6) admits therefore a factorization $F_M(M) \to L \xrightarrow{a} C$. By construction of C (2.7) this implies that a is an epimorphism. As every epimorphism with a left inverse is an isomorphism, the theorem follows. $\square$

2.15. Now assume that $\mathcal{C}$ is itself a monoidal category, that $\mathcal{M}$ is a symmetric monoidal category and that the monoidal structure of $\mathcal{C}$ is compatible with the action of $\mathcal{M}$, i.e., there are given isomorphisms

$$\alpha_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M),$$
$$\sigma_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} (X \otimes M) \otimes Y$$

functorial in $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $M \in \mathrm{Ob}(\mathcal{M})$ such that they are compatible with the associativity and unit constraints in $\mathcal{C}$ and $\mathcal{M}$ and the commutativity constraint in $\mathcal{M}$ and such that for X, Y in $\mathcal{C}$ and N, M in $\mathcal{M}$ we have $\sigma_{X,Y,M \otimes N} = \sigma_{X \otimes M, Y, N} \circ (\sigma_{X,Y,M} \otimes \mathrm{id}_N)$ and the following diagram is commutative:

$$\begin{tikzcd} (X \otimes M) \otimes (Y \otimes N) & (X \otimes (Y \otimes N)) \otimes M \arrow[l, "\sigma_{X, Y \otimes N, M}"'] & ((X \otimes Y) \otimes N \otimes M \arrow[l, "\alpha_{X,Y,N} \otimes \mathrm{id}_M"'] \\ ((X \otimes M) \otimes Y) \otimes N \arrow[u, "\alpha_{X \otimes M, Y, N}"'] & ((X \otimes Y) \otimes M) \otimes N \arrow[l, "\sigma_{X,Y,M} \otimes \mathrm{id}_N"'] \arrow[r, phantom, "\cong"] & (X \otimes Y) \otimes M \otimes N. \arrow[u, "\text{comm.constr.}"] \end{tikzcd}$$

Then we have a canonical morphism of functors

$$\begin{aligned} \omega_2 \otimes \omega_2 &\to \big(\omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2)\big) \otimes \big(\omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2)\big) \\ &\cong \omega_1 \otimes \omega_1 \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2), \end{aligned}$$

which induces a morphism

$$\mu : \underline{\mathrm{CoHom}}(\omega_1 \otimes \omega_1, \omega_2 \otimes \omega_2) \to \underline{\mathrm{CoHom}}(\omega_1, \omega_2) \otimes \underline{\mathrm{CoHom}}(\omega_1, \omega_2). \quad (2.15.1)$$

2.16. By applying 2.7 to ω and $\omega\otimes\omega$, it follows that $\underline{\mathrm{CoEnd}}(\omega)$ and $\underline{\mathrm{CoEnd}}(\omega\otimes\omega)$ are corepresentable if ω factors through a monoidal subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that the action of $\mathcal{M}$ on $\mathcal{C}$ is coclosed for $\mathcal{C}'$ and if in $\mathcal{M}$ exist small inductive limits.

2.17. We keep the notations of 2.15 and we assume that $\underline{\mathrm{CoEnd}}(\omega)$ and $\underline{\mathrm{CoEnd}}(\omega\otimes\omega)$ are corepresentable and that (2.15.1) is an isomorphism. Let $\mathcal{D}$ be also a monoidal category and let $\omega:\mathcal{D}\to\mathcal{C}$ be a tensor functor. We get a multiplication as the composition

$$\begin{aligned}\underline{\mathrm{CoHom}}(\omega_1,\omega_2)\otimes\underline{\mathrm{CoHom}}(\omega_1,\omega_2) &\xrightarrow{\mu^{-1}} \underline{\mathrm{CoHom}}(\omega_1\otimes\omega_1,\omega_2\otimes\omega_2)\\ &= \underline{\mathrm{CoHom}}(\omega_1\circ\otimes,\omega_2\circ\otimes)\\ &\to \underline{\mathrm{CoHom}}(\omega_1,\omega_2),\end{aligned}\tag{2.17.1}$$

where the last morphism is given by 2.9.

2.18. Now assume that $\mathcal{C}=\mathcal{M}$ with the canonical right action and that $\underline{\mathrm{CoHom}}(\omega_1,\omega_2)$ and $\underline{\mathrm{CoHom}}(\omega_1\otimes\omega_1,\omega_2\otimes\omega_2)$ are corepresentable (2.8). Then (2.15.1) is an isomorphism [7, 2.3]. Further as in loc. cit. 2.4–2.9 we have:

Proposition. *The product* 2.17 *makes* $\underline{\mathrm{CoHom}}(\omega_1,\omega_2)$ *into a monoid in* $\mathcal{M}$. *If* $\mathcal{M}$ *and* $\mathcal{D}$ *are braided* (*respectively symmetric*) *monoidal categories,* $\underline{\mathrm{CoHom}}(\omega_1,\omega_2)$ *is a dual quasitriangular* (*loc. cit.* 2.8) (*respectively commutative*) *monoid. If* $\omega_1=\omega_2=\omega$, *we further have*:

(1) *The comultiplication and counit of* $C=\underline{\mathrm{CoEnd}}(\omega)$ *are homomorphisms of monoids, i.e.,* C *is a bimonoid and* ω *factorizes in*

$$\mathcal{D}\xrightarrow{\omega^C}\mathcal{M}^C\xrightarrow{\text{forget}}\mathcal{M}.$$

(2) *If* L *is any other bimonoid in* $\mathcal{M}$ *such that* ω *factorizes through a tensor functor* $\mathcal{D}\to\mathcal{M}^L$ *and the forgetful functor* $\mathcal{M}^L\to\mathcal{M}$, *then there exists a unique homomorphism of bimonoids* $C\to L$ *such that* ω *factorizes through the induced functor* $\mathcal{M}^C\to\mathcal{M}^L$.

(3) *If* $\mathcal{D}$ *is rigid* C *has an antipode.*

2.19. We remark that the very restrictive assumptions on $\mathcal{M}$ made in loc. cit. 2.2 (namely that $\mathcal{M}$ is rigid and has arbitrary inductive limits) are needed only to ensure the representability of $\underline{\mathrm{CoHom}}(\omega)$ and $\underline{\mathrm{CoHom}}(\omega\otimes\omega)$ which follows here from 2.8.

2.20. Corollary. *Let* $\mathcal{M}$ *be a braided monoidal category and let* L *be a Hopf monoid in* $\mathcal{M}$. *Denote by* $\mathcal{D}$ *the monoidal subcategory of rigid objects in* $\mathcal{M}$ *and by* $\omega:\mathcal{D}^L\to\mathcal{M}$ *the canonical functor. Assume that* 2.13(b)–(d) *hold. Then* $\underline{\mathrm{CoEnd}}(\omega)=L$ *as Hopf monoids.*

Proof. It is easy to see that $\mathcal{D}^L$ is again rigid because L admits an antipode. Therefore, the corollary follows from 2.8, 2.14, and 2.18. □

3. Additive monoidal categories

3.1. Let R be a commutative ring. A monoidal category $\mathcal{M}$ is called *R-linear* if the underlying category is R-linear and if $\otimes$ is an R-bilinear functor. An R-linear monoidal category is called *pseudoabelian* (respectively *abelian*), if the underlying R-linear category is pseudoabelian (respectively abelian).

3.2. If $\mathcal{M}$ is an R-linear monoidal category and if $\varphi : A \to R$ is a homomorphism of commutative rings, $\mathcal{M}$ is also an A-linear monoidal category. We call this the *underlying A-linear monoidal category* and write also $\varphi_*\mathcal{M}$ if we consider $\mathcal{M}$ as an A-linear tensor category via φ.

Conversely, let $\mathcal{M}$ be an additive monoidal category. Then $R = \mathrm{End}(1)$ is a ring. For every object X of $\mathcal{M}$ the action of R an X induced by $X \xrightarrow{\sim} 1 \otimes X$ is equal to the action of R on X induced by $X \xrightarrow{\sim} X \otimes 1$. In particular, R is commutative and the category $\mathcal{M}$ gets the structure of an R-linear monoidal category. Let us denote by $\mathcal{M}_{/R}$ this R-linear monoidal category. The ring R has the following universal property. Let A be a commutative ring and let $\mathcal{M}_{/A}$ be an A-linear monoidal category such that the underlying additive monoidal category is $\mathcal{M}$. Then there exists a unique ring homomorphism $\varphi : A \to R$ such that $\varphi_*(\mathcal{M}_{/R}) = \mathcal{M}_{/A}$. Indeed, $R = \mathrm{End}_{\mathcal{M}_{/A}}(1)$ is an A-algebra and this defines φ.

3.3. Let $\mathcal{C}$ be an R-linear category and let $\mathcal{M}$ be an R-linear monoidal category acting on $\mathcal{C}$ from the left. We call this action *R-bilinear* if the functor $\mathcal{M} \times \mathcal{C} \to \mathcal{C}$ is R-bilinear.

3.4. Let $\varphi : R \to R'$ be a homomorphism of commutative rings, and let $\mathcal{C}$ be an R-linear category. Then the category $\mathcal{C}_{R'}$ *obtained from $\mathcal{C}$ by scalar extension* φ is defined as follows. The objects are the same as the objects of $\mathcal{C}$ and for two objects X and Y in $\mathcal{C}_{R'}$ define

$$\mathrm{Hom}_{\mathcal{C}_{R'}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_R R'.$$

This way we get an R'-linear category which is denoted by $\mathcal{C}_{R'}$.

3.5. We have an obvious R-linear functor

$$i_{R'} : \mathcal{C} \to \mathcal{C}_{R'},$$

which is bijective on objects.

If $\mathcal{C}'$ is an R'-linear category and $F : \mathcal{C} \to \mathcal{C}'$ is an R-linear functor, F factorizes in $F' \circ i_{R'}$ where $F' : \mathcal{C}_{R'} \to \mathcal{C}'$ is an R'-linear functor which is uniquely determined.

3.6. Let $\varphi : R \to R'$ be flat, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then if f is a monomorphism (respectively an epimorphism) in $\mathcal{C}$ its image in $\mathcal{C}_{R'}$ is a monomorphism (respectively epimorphism) in $\mathcal{C}_{R'}$. The converse holds if R' is faithfully flat over R.

3.7. Let $\mathcal{D}$ be a second R-linear category and let $\omega : \mathcal{C} \to \mathcal{D}$ be an R-linear functor. Then ω induces a functor $\omega_{R'} : \mathcal{C}_{R'} \to \mathcal{D}_{R'}$. If ω is fully faithful, so is $\omega_{R'}$. Further, if R' is flat over R and if ω is faithful, then $\omega_{R'}$ is also faithful. The converse holds if R' is faithfully flat over R.

3.8. Now assume that $\mathcal{M}$ is a monoidal R-linear category. The R-bilinear functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ extends to an R'-bilinear functor $\otimes : \mathcal{M}_{R'} \times \mathcal{M}_{R'} \to \mathcal{M}_{R'}$. This way $\mathcal{M}_{R'}$ is a monoidal R'-linear category. It is symmetric (respectively braided, respectively rigid) if $\mathcal{M}$ is symmetric (respectively braided, respectively rigid).

3.9. Let $\mathcal{T}$ be an R-linear monoidal category. Then there exists an R-linear pseudo-abelian monoidal hull $\mathcal{T}'$. Its underlying category is the pseudo-abelian hull of the underlying additive category, i.e., the objects of $\mathcal{T}'$ are pairs (X, p) where $p \in \operatorname{End}(X)$ is a projector and we set $\operatorname{Hom}((X, p), (Y, q)) = q \operatorname{Hom}(X, Y) p$. We set $(X, p) \otimes (Y, q) := (X \otimes Y, p \otimes q)$. The unit in $\mathcal{T}'$ is defined as $(1, \mathrm{id}_1)$. As associativity, left unit, and right unit constraint are functorial we get induced constraints on $\mathcal{T}'$. This defines the structure of a monoidal category on $\mathcal{T}'$. The same argument applied to a commutativity constraint shows that $\mathcal{T}'$ is symmetric if and only if $\mathcal{T}$ is symmetric. The canonical $\otimes$-functor $\mathcal{T} \to \mathcal{T}'$ induces $\operatorname{End}(1_{\mathcal{T}}) = \operatorname{End}(1_{\mathcal{T}'})$, in particular 3.2, $\mathcal{T}'$ is again R-linear, and $\otimes$ is R-bilinear. Further, $\mathcal{T}$ is rigid if and only if $\mathcal{T}'$ is rigid. Indeed, if X admits a dual $X^\vee$, the dual of (X, p) is given by $(X^\vee, p^\vee)$ where $p^\vee$ denotes the transpose of p.

4. Groupoids and gerbes

4.1. Let $\mathcal{S} = (\mathtt{Sch}/S)$ be the site of schemes over some scheme S equipped with the *fpqc*-topology.

A stack in groupoids $\mathcal{G}$ over $\mathcal{S}$ is called a *gerbe* if the following two conditions are satisfied:

(a) $\mathcal{G}$ is locally nonempty, i.e., there exists a covering $(U_i \to S)$ in $\mathcal{S}$ such that the fibre categories $\mathcal{G}_{U_i}$ are nonempty.
(b) $\mathcal{G}$ is locally connected, i.e., for every object T in $\mathcal{S}$ and for all objects $x, y \in \mathcal{G}_T$ there exists a covering $(V_i \to T)$ such that $\operatorname{Hom}(x|_{V_i}, y|_{V_i})$ is nonempty.

A gerbe $\mathcal{G}$ over $\mathcal{S}$ is called *neutral* if it is globally nonempty, i.e., if $\mathcal{G}_S$ is nonempty. If G is a sheaf of groups in the topos of $\mathcal{S}$, the fibered category $\mathtt{Tors}(G)$ whose fibre over an object T of $\mathcal{S}$ is the category of right G-torsors on T is a neutral gerbe.

Conversely, let $\mathcal{G}$ be a neutral gerbe, and let x be an object in $\mathcal{G}_S$. Set $G = \operatorname{Aut}(x)$. By definition this is a group in the topos of $\mathcal{S}$. Then

$$\mathcal{G}_T \to \mathtt{Tors}(T, G), \quad y \mapsto \underline{\operatorname{Isom}}(x_T, y)$$

is an equivalence of $\mathcal{G}$ with $\mathtt{Tors}(G)$.

4.2. An *S-groupoid acting on an S-scheme X* is a scheme G over S equipped with S-morphisms $t, s : G \to X$ and a composition law $\circ : G_s \times_{X,t} G \to G$ which is a scheme morphism over $X \times X$ such that for every S-scheme T the data

$$X(T), \qquad G(T), \qquad t, s : G(T) \to X(T), \qquad \circ : G(T) \times G(T) \to G(T)$$

define a category (where $X(T)$ is the set of objects, $G(T)$ the set of morphisms, t (respectively s) the target (respectively source), and $\circ$ the composition law) in which every morphism is invertible.

The identity in the morphisms sets defines a morphism of $X \times X$-schemes $\varepsilon : X \to G$ (X diagonally embedded in $X \times X$).

Let $(G', t', s', \circ')$ be a second S-groupoid acting on X. A *homomorphism $G \to G'$ of groupoids* is an $X \times_S X$-morphism $G \to G'$ which is compatible with $\circ$ and $\circ'$ in the obvious sense. We denote by $\mathtt{Grpd}_S(X)$ the category of S-groupoids acting on X.

4.3. Let G be an S-groupoid acting on an S-scheme X. For every morphism of schemes $u : Y \to X$ the inverse image $u^*(G) = G_Y$ is defined by the Cartesian diagram

$$\begin{tikzcd} G_Y \arrow[r] \arrow[d] & G \arrow[d, "s \times t"] \\ Y \times_S Y \arrow[r, "u \times u"] & X \times_S X. \end{tikzcd}$$

This way we get a fibered category $\mathtt{Grpd}_S$ over $(\mathtt{Sch}/S)$.

4.4. Let X be an S-scheme and let $p : G \to X$ be an X-group scheme. If we set $t = s = p$, then the morphism $(t, s) : G \to X \times_S X$ factorizes through the diagonal $\Delta : X \to X \times_S X$. The group multiplication $\circ : G \times_X G \to G$ is therefore a morphism of schemes over $X \times_S X$, and the data $(G, t, s, \circ)$ define a groupoid.

Conversely, if G is a groupoid acting on an S-scheme X such that $s = t$, i.e., $(t, s) : G \to X \times_S X$ factorizes through the diagonal, then the data $(G, s = t, \circ)$ define a group scheme over X. Therefore, we can identify the fibered category $\mathtt{Gr}_S$ of group schemes over various S-schemes with a full fibred subcategory of the fibered category of groupoids.

In particular, every groupoid acting on S is a group scheme over S.

4.5. For a groupoid G acting on X, the fibre product of

$$\begin{tikzcd} & G \arrow[d, "(t,s)"] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

is a group scheme over X which we will denote by G^Δ. This construction defines a $(\mathtt{Sch}/S)$-functor of fibered categories $\mathtt{Grpd}_S \to \mathtt{Gr}_S$ which is right adjoint to the inclusion $\mathtt{Gr}_S \to \mathtt{Grpd}_S$ (it suffices to show that for every S-scheme X the induced functor

$\mathtt{Grpd}_S(X) \to \mathtt{Gr}_S(X)$ is left adjoint to the inclusion $\mathtt{Gr}_S(X) \to \mathtt{Grpd}_S(X)$ and this is obvious).

4.6. Note that if G is a group scheme over an S-scheme X, the pull back $u^*(G)$ via a morphism $u : Y \to X$ (4.3) is in general no group scheme over Y. But we have $u^*(G)^{\Delta} = G \times_X Y$.

4.7. Let $(G, t, s, \circ)$ be an S-groupoid acting on an S-scheme X. Let $S' \to S$ be a morphism of schemes. Define $G' = G \times_S S'$, $X' = X \times_S S'$, $t' = t_{S'}$, $s' = s_{S'}$, and $\circ' = \circ_{S'}$ where we identify $(G \times_X G) \times_S S' = (G \times_S S') \times_{X \times_S S'} (G \times_S S')$. Then $(G', t', s', \circ')$ is an S'-groupoid acting on X'. It is called the *base change of* $(G, t, s, \circ)$ *by* $S' \to S$.

4.8. The groupoid G acts *transitively* on X (with respect to the *fpqc*-topology) if there exists a *fpqc*-covering $(T \to X \times_S X)$ such that $\mathrm{Hom}_{X \times_S X}(T, G) \neq \emptyset$.

If $u : Y \to X$ is an S-scheme morphism and if $G_T = u^*(G)$ is the inverse image of G, G_T acts transitively on T.

4.9. Lemma. *Let $G \xrightarrow{p} X$ be a group scheme over some S-scheme X, which we consider as an S-groupoid acting on X. Then G acts transitively if and only if $X \to S$ is a monomorphism.*

Proof. The morphism $X \to S$ is a monomorphism if and only if the diagonal $\Delta : X \to X \times_S X$ is an isomorphism. Therefore the condition is obviously sufficient. Let $c : T \to X \times_S X$ be a faithfully flat quasi-compact morphism such that there exists an $(X \times_S X)$-morphism $v : T \to G$. We get a commutative diagram

$$\begin{tikzcd} T \arrow[r, "v"] \arrow[d, "c"'] & G \arrow[d, "p"] \\ X \times_S X & X. \arrow[l, "\Delta"'] \end{tikzcd}$$

In particular, c factorizes through Δ. Therefore Δ is a closed surjective immersion. We have to show that the defining ideal $\mathcal{I}$ of Δ is zero. Let $V \subset X \times_S X$ be some open subset and x be a section of $\mathcal{I}$ over V. Then its image under $c^{\#} : \Gamma(V, \mathcal{O}_{X \times X}) \to \Gamma(c^{-1}(V), \mathcal{O}_T)$ is zero. But this map is injective because c is faithfully flat. Therefore x is zero. $\square$

4.10. Let $S = \mathrm{Spec}(R)$ be affine and let G be an S-groupoid acting on an affine S-scheme $X = \mathrm{Spec}(B)$. Assume that G is affine over $X \times_S X$, say $G = \mathrm{Spec}(L)$. Then L is a $B \otimes_R B$-module via (t, s), i.e., a (B, B)-bimodule such that the two induced R-module structures coincide. We write the B-module structure induced by t (respectively by s) as left (respectively right) B-module structure. Further, the (B, B)-bimodule has the following additional structures:

(a) L is a commutative $B \otimes_R B$-algebra and admits therefore a product

$$m : L \otimes_{B \otimes_R B} L \to L.$$

(b) The composition law $G \times_X G \to G$ corresponds to a $B \otimes_R B$-algebra homomorphism

$$c : L \to L \otimes_B L,$$

the identity $\varepsilon : X \to G$ corresponds to a $B \otimes_R B$-algebra homomorphism

$$e : L \to B,$$

and the inversion morphism $G \to G$ defines an antipode

$$S : L \to L.$$

This way L obtains the structure of a Hopf monoid (1.6) in the category of (B, B)-bimodules. Conversely, every Hopf monoid in the category of (B, B)-bimodules defines an affine groupoid G acting on X.

4.11. Let G be an S-groupoid acting on an S-scheme X. For every S-scheme T we have a category $\mathcal{G}^0_T = (X(T), G(T), \circ)$ in which all morphisms are isomorphisms. This categories form a fibered category $\mathcal{G}^0 = \mathcal{G}^0_{X:G}$ over the category of S-schemes. The inverse image functors are given as follows: Let $u : T' \to T$ be a morphism of S-schemes. The inverse image of an object x of $\mathcal{G}^0_T$ (that is of an element $x \in X(T)$) is the composition $x \circ u$. The inverse image of a morphism f of $\mathcal{G}^0_T$ (that is of an element $f \in G(T)$) is the composition $f \circ u$.

If $x, y \in X(T)$ are two objects of $\mathcal{G}^0_T$ the functor on Sch/T which associates to $u : T' \to T$ the set $\mathrm{Hom}_{\mathcal{G}^0_{T'}}(u^*x, u^*y)$ is representable by the fibre product of

$$\begin{tikzcd} & G \arrow[d, "{(b,s)}"] \\ T' \arrow[r, "{(x,y)\circ u}"] & S \times S \end{tikzcd}$$

In particular, it is a sheaf for the *fpqc*-topology, and $\mathcal{G}^0_{X:G}$ is a pre-stack for the *fpqc*-topology. Denote by $\mathcal{G} = \mathcal{G}_{X:G}$ the associated stack.

4.12. Proposition. *Let X be an S-scheme such that $X \to S$ is a fpqc-covering and let G be a groupoid which acts on X. Then the stack $\mathcal{G}_{X:G}$ is a gerbe if and only if the action of G on X is transitive.*

If in this case G is a group scheme over X, then $X \to S$ is an isomorphism.

Proof. As $X \to S$ is a covering the stack $\mathcal{G}_{X:G}$ is locally nonempty, and by definition the action of G on X is transitive if and only if $\mathcal{G}_{X:G}$ is locally connected. The last assertion follows from 4.9 and the following lemma.

Lemma. *Let $f : X \to S$ be a faithfully flat quasicompact monomorphism of schemes. Then f is an isomorphism.*

Proof. By faithfully flat descent, the morphism f is an isomorphism if and only if $f \times \mathrm{id}_X$ (base change of f by f) is an isomorphism. But this is the second projection $X \times_S X \to X$ which is an isomorphism, because f is a monomorphism. □

4.13. The construction above is functorial in the following sense. Let $u : G_1 \to G_2$ be a morphism of groupoids acting on an S-scheme X. Then for every S-scheme T, u defines a functor $\mathcal{G}_{X:G_1}(T) \to \mathcal{G}_{X:G_2}(T)$ by inducing the identity on objects and by sending a morphism $a \in G_1(T)$ to $u \circ a \in G_2(T)$. It is easy to check that this gives indeed a $(\mathtt{Sch}/S)$-functor $\mathcal{G}^0_{X:G_1} \to \mathcal{G}^0_{X:G_2}$ and therefore also a fibered functor of the associated stacks.

4.14. Proposition. *Let $u : \mathcal{G}_1 \to \mathcal{G}_2$ be a $(\mathtt{Sch}/S)$-functor between gerbes on $(\mathtt{Sch}/S)$. Let X be an S-scheme. For some object ω in $\mathcal{G}_{i,X}$ let $\underline{\mathrm{Aut}}_X(\omega)$ be the sheaf over X which associates to $p : T \to X$ the set of automorphisms of $p^*\omega$ in $\mathcal{G}_{i,T}$.*

Assume that $X \to S$ is a fpqc-covering such that there exists an object ω in $\mathcal{G}_{1,X}$ and assume that $u : \underline{\mathrm{Aut}}_X(\omega) \to \underline{\mathrm{Aut}}_X(u(\omega))$ is an isomorphism. Then ω is an equivalence.

Proof. This follows from [6, Chapter IV, 2.2.6(iii)]. □

4.15. Corollary. *Let G_1 and G_2 be two S-groupoids acting transitively on a fpqc-covering $X \to S$ and let $u : G_1 \to G_2$ be a morphism of groupoids. Then u is an isomorphism if and only if $u^\Delta : G_1^\Delta \to G_2^\Delta$ is an isomorphism.*

Proof. The group scheme G_i^Δ represents the functor $\underline{\mathrm{Aut}}_X(\mathrm{id}_X)$ and we conclude by 4.14. □

4.16. Let $\mathcal{G}$ be a gerbe over $(\mathtt{Sch}/S)$. If for every S-scheme X and for $\omega_1, \omega_2 \in \mathcal{G}(X)$ the *fpqc*-sheaf $\underline{\mathrm{Isom}}_X(\omega_1, \omega_2)$ if representable by a scheme which is affine over X we say that $\mathcal{G}$ is *affinely tied*.

As a gerbe is by definition locally connected, it is affinely tied if $\underline{\mathrm{Isom}}_X(\omega_1, \omega_2) \to X$ is representable and affine for one *fpqc*-covering $X \to S$ and for one choice $\omega_1, \omega_2 \in \mathcal{G}(X)$.

4.17. Denote by $\mathtt{Cov}(S)$ the full subcategory of $(\mathtt{Sch}/S)$ which are a *fpqc*-coverings of S. Define a fibered category $\mathtt{AffGerb}$ over $\mathtt{Cov}(S)$: a fibre over a covering $X \to S$ consists of pairs $(\mathcal{G}, \omega)$ where $\mathcal{G}$ is an affinely tied gerbe over $(\mathtt{Sch}/S)$ and where $\omega \in \mathcal{G}(X)$. The inverse image functors are given by pulling back ω. A morphism $(\mathcal{G}, \omega) \to (\mathcal{G}', \omega')$ in $\mathtt{AffGerb}(X)$ is a morphism of gerbes over $(\mathtt{Sch}/S)$ sending ω to ω'.

On the other hand, denote by $\mathtt{AffGrpd}$ the fibered category over $\mathtt{Cov}(S)$ whose fibre over a covering $X \to S$ consists of groupoids G acting transitively on X and which are affine over $X \times_S X$.

Then we get an equivalence of fibred categories $\mathtt{AffGerb} \approx \mathtt{AffGrpd}$ via

$$G = \underline{\mathrm{Isom}}_{X \times_S X}\bigl(\mathrm{pr}_2^* \omega, \mathrm{pr}_1^* \omega\bigr) \xrightarrow{(t,s)} X \times_S X,$$
$$\mathcal{G} = \mathcal{G}_{X:G}, \qquad \omega = \mathrm{id}_X \in \mathcal{G}_{X:G}(X).$$

If G is a groupoid acting on a cover X which is an X-group scheme, then $X \to S$ is an isomorphism and its image in $\mathtt{AffGerb}$ is isomorphic to a pair consisting of a neutral gerbe and an element $\omega \in \mathcal{G}(S)$.

4.18. Let G be an S-groupoid acting on an S-scheme X. A *representation of G* is a quasi-coherent $\mathcal{O}_X$-module M together with an action ρ of G, i.e., for every S-scheme T and for every $g \in G(T)$ there is a morphism $\rho(g) : s(g)^*(M) \to t(g)^*(M)$ between the inverse images of M under $s(g)$ and $t(g) : T \to X$. These morphisms are supposed to be compatible with base change $T' \to T$, to satisfy $\rho(gh) = \rho(g)\rho(h)$ (for $s(g) = t(h)$), and such that for $g = \mathrm{id}_x = \varepsilon(x)$ with $x \in X(T)$ the homomorphism $\rho(g)$ is the identity of $x^*(M)$. As G is a groupoid the homomorphisms $\rho(g)$ are automorphisms.

Let $\mathtt{Rep}(X{:}G)$ be the category of finite locally free $\mathcal{O}_X$-modules equipped with an action of G. Together with the obvious symmetric monoidal structure it is a rigid symmetric monoidal category. If G acts transitively on $X = S$, G is a group scheme and we get the category $\mathtt{Rep}(G)$ of representations on finite locally free $\mathcal{O}_S$-modules.

4.19. Let $\mathcal{F}$ be some fibered category over $(\mathtt{Sch}/S)$. A *representation of $\mathcal{F}$* is a $(\mathtt{Sch}/S)$-functor of $\mathcal{F}$ into the stack of quasi-coherent sheaves over $\mathtt{Sch}/S$ which is compatible with base change. We write $\mathtt{Rep}(\mathcal{F})$ for the category of representations of $\mathcal{F}$.

If $\mathcal{F}^0$ is a pre-stack over $\mathtt{Sch}/S$ with associated stack $\mathcal{F}$, the universal property of $\mathcal{F}$ implies $\mathtt{Rep}(\mathcal{F}^0) = \mathtt{Rep}(\mathcal{F})$.

4.20. Let G be an S-groupoid acting on an S-scheme X and let R be a representation of the fibered category $\mathcal{G}^0_{X:G}$. For every S-scheme T and for every S-morphism $x : T \to X$ in $\mathcal{G}^0_{X:G}(T)$ we have an isomorphism

$$R(x) \xrightarrow{\sim} x^* R(\mathrm{id}_X),$$

and R is determined by the quasicoherent $\mathcal{O}_X$-module $R_0 = R(\mathrm{id}_X)$ and by the $R(g) : x^* R_0 \to y^* R_0$ for $g : x \to y$ in $\mathcal{G}^0_T$. These $R(g)$ form a representation of the groupoid G on R_0 and we get an equivalence

$$\mathtt{Rep}(\mathcal{G}_{X:G}) = \mathtt{Rep}\bigl(\mathcal{G}^0_{X:G}\bigr) \approx \mathtt{Rep}(X{:}G).$$

4.21. Let G be an S-groupoid which acts transitively on an S-scheme X, let $u: Y \to X$ be an S-morphism and denote by $G_Y = u^*G$ the pullback of G which acts transitively on Y. Suppose that $X \to S$ and $Y \to S$ are coverings. The morphisms of pre-stacks $u: \mathcal{G}^0_{Y:G_Y} \to \mathcal{G}^0_{X:G}$ induces an isomorphism of the automorphism sheaf of id_Y in $\mathcal{G}^0_{Y:G_Y}$ with the sheaf of automorphisms of u in $\mathcal{G}^0_{X:G}$. The induced morphisms of gerbs

$$u: \mathcal{G}_{Y:G_Y} \to \mathcal{G}_{X:G}$$

is therefore an equivalence (4.14). In particular, we get

$$\mathtt{Rep}(X:G) \approx \mathtt{Rep}(Y:G_Y).$$

5. Reconstruction of groupoids over Prüfer rings

5.1. In this section we denote by R a (commutative) ring and by B a unital R-algebra (not necessarily commutative). If R' is a second ring and B' a unital R'-algebra a morphism $(R, B) \to (R', B')$ is by definition a pair (ψ, φ) where $\psi: R \to R'$ is a homomorphism of rings and $\varphi: B \to B'$ is a homomorphism of R-algebras.

Denote by $\mathcal{M}$ the category of (B, B)-bimodules such that the two underlying R-module structures coincide. Tensorizing over B endows $\mathcal{M}$ with the structure of an R-linear monoidal category. The 1 is given by the (B, B)-bimodule B. Less symmetrically, we can $\mathcal{M}$ also consider as the category of right $(B^{\mathrm{opp}} \otimes_R B)$-modules.

Denote by $\mathcal{C}$ the category of right B-modules. For every right B-module X and every (B, B)-bimodule L, $X \otimes_B L$ is again a right B-module and this defines a right action of $\mathcal{M}$ on $\mathcal{C}$.

Following Deligne [3, 1.15], we call a comonoid in the monoidal category $\mathcal{M}$ an *R-coalgebroid acting on B*. If L is an R-coalgebroid acting on B we call an L-comodule in $\mathcal{C}$ (1.9) simply an *L-comodule over B*.

5.2. Let L be an R-coalgebroid acting on B. Assume that L is flat as a left B-module. A comodule homomorphism is a monomorphism (respectively an epimorphism) if and only if it is injective (respectively surjective). It follows that the category of L-comodules is abelian and the functor “forgetting the coaction” is exact.

5.3. Let (L, c, e) be an R-coalgebroid acting on B. We want to define the base change of (L, c, e) with respect to a morphism $(R, B) \to (R', B')$. We do this in three steps.

(1) Let B' be an R-algebra, and let $\varphi: B \to B'$ be a homomorphism of R-algebras. Then $L' = B' \otimes_B L \otimes_B B'$ is a (B', B')-bimodule. Define a comultiplication c' as the composition

$$\begin{aligned} B' \otimes_B L \otimes_B B' &\xrightarrow{1\otimes c\otimes 1} B' \otimes_B L \otimes_B L \otimes_B B' \\ &\longrightarrow \big(B' \otimes_B L \otimes_B B'\big) \otimes_{B'} B' \otimes_B L \otimes_B B', \end{aligned}$$

where the second arrow sends $b'_1 \otimes l_1 \otimes l_2 \otimes b'_2$ to $b'_1 \otimes l_1 \otimes 1 \otimes 1 \otimes l_2 \otimes b'_2$. Further define a counit e' as the composition

$$B' \otimes_B L \otimes_B B' \xrightarrow{1 \otimes e \otimes 1} B' \otimes_B B' \to B',$$

where the second arrow is given by the multiplication in B'. It is straightforward to see that (L', c', e') is an R-coalgebroid acting on B'. We denote it by $\varphi^*(L, c, e)$ or simply $\varphi^*(L)$.

(2) Now let $R \to R'$ be a homomorphism of commutative rings and set $B' = B \otimes_R R'$. Then $L \otimes_R R'$ is a (B', B')-bimodule such that the underlying R'-module structures coincide. Further, $c \otimes \mathrm{id}_{R'}$ defines a comultiplication on $L \otimes_R R'$ if we identify $(L \otimes_R R') \otimes_{B \otimes_R R'} (L \otimes_R R')$ with $(L \otimes_B L) \otimes_R R'$. Then $(L \otimes_R R', c \otimes \mathrm{id}_{R'}, e \otimes \mathrm{id}_{R'})$ is an R'-coalgebroid acting on B' which we denote by $(L, c, e)_{R'}$ or simply $L_{R'}$.

(3) Now consider the general situation. Let $R \to R'$ be a homomorphism of commutative rings, let B' be an R'-algebra, and let $\varphi : B \to B'$ be a homomorphism of R-algebras. Then φ induces a R'-algebra homomorphism $\psi : B \otimes_R R' \to B'$ and $\psi^*(L_{R'})$ is an R'-coalgebroid acting on B' which we will also denote simply by $\varphi^*(L)$. The underlying (B', B')-bimodule is given by

$$\left(B' \otimes_B L \otimes_B B'\right) \otimes_{R' \otimes_R R'} R' \xrightarrow{\sim} B' \otimes_{B \otimes_R R'} \left(L \otimes_R R'\right) \otimes_{B \otimes_R R'} B'.$$

5.4. Denote by $\mathcal{D}$ the full subcategory of $\mathcal{C}$ of B-modules which are finitely generated projective. Then the action of $\mathcal{M}$ on $\mathcal{C}$ is coclosed for $\mathcal{D}$ (2.5). Indeed, if M is a finitely generated projective right B-module, the functor

$$F_M : \mathcal{C} \to \mathcal{M}, \quad N \mapsto M^\vee \otimes_R N$$

is left adjoint to $(M \otimes_B)$ and F_M depends functorially (and contravariantly) on M.

5.5. Now let L be an R-coalgebroid acting on B which is flat over B for both B-module structures. We now want to apply 2.14 to the forgetful functor $\omega : \mathcal{D}^L \to \mathcal{C}$ (for a special class of rings B). For this we have to check that the assumptions in 2.13 hold. First, forgetting the left action defines a functor $\Phi : \mathcal{M} \to \mathcal{C}$. This functor is faithful. Further, it preserves and reflects filtered inductive limits because this holds for any functor which forgets an algebraic structure (e.g., [12, 18.5.3]). Therefore 2.13(e) holds.

Further, $\mathcal{D}$ is equivalent to a small category and therefore this holds for $\mathcal{D}^L$ as well, and in $\mathcal{M}$ exist inductive limits. Therefore by 5.4 the assumption 2.13(a) holds. Further 2.13(b) and (c) are clear. It remains to check (d).

5.6. Let L be an R-coalgebroid acting on B. Assume that L is flat as a left B-module. If (M, r) is an L-comodule we call a subset $N \subset M$ an *L-subcomodule* if N is a B-submodule and if $r(N) \subset N \otimes_B L$ (note that because of the flatness of L we can consider $N \otimes_B L$ as a subset of $M \otimes_B L$).

The intersection of L-subcomodules is again an L-subcomodule. In particular, for every subset S of M there exists a smallest L-subcomodule containing S which will be called the *L-comodule generated by S*.

5.7. Proposition (cf. [13]). *Let L be an R-coalgebroid acting on B such that L is flat as a left B-module. Let (M, r) be an L-comodule and let E be a B-submodule of L which is finitely generated. Then there exists an L-subcomodule N containing E which is also finitely generated as a B-module.*

Proof. It suffices to show the following:

Lemma. *Let m be an element of M and let N be the L-subcomodule generated by $\{m\}$. Then N is finitely generated as a right B-module.*

Proof. Write

$$r(m) = \sum_{i=1}^{d} n_i \otimes a_i$$

with $n_i \in N$ and $a_i \in L$ and let N' be the B-module generated by the n_i. We claim that $N' = N$. We have to show that $N \subset N'$. We set $E = r^{-1}(N' \otimes L) \subset M$. Using (COM2) we see that $E \subset N'$ and by definition $m \in E$. Therefore it suffices to show that E is an L-subcomodule of M, i.e., $r(E) \subset E \otimes L$. As L is flat we have $E \otimes L = (r \otimes \mathrm{id}_L)^{-1}(N' \otimes L \otimes L)$. By (COM1), we have

$$(r \otimes \mathrm{id}_L)\bigl(r(E)\bigr) = (\mathrm{id}_M \otimes c)\bigl(r(E)\bigr) \subset (\mathrm{id}_M \otimes c)\bigl(N' \otimes L\bigr) \subset N' \otimes L \otimes L$$

and therefore E is an L-subcomodule. □

5.8. Corollary. *Let L be an R-coalgebroid acting on B such that L is flat as a left B-module. Then every L-comodule is the filtered union of L-subcomodules which are finitely generated as B-modules.*

5.9. Let B be a commutative integral domain. Recall that B is called *Prüfer ring* if the following equivalent conditions hold:

(a) Every localization of B at a prime ideal is a valuation ring.
(b) Every finitely generated submodule of a flat B-module is projective.

Further, a module over R is flat iff it is torsionfree. In particular, every submodule of a flat module is again flat. A noetherian Prüfer ring is a Dedekind ring.

5.10. Corollary. *Let B be a Prüfer ring and let L be an R-coalgebroid acting on B such that L is flat as a left B-module. Then every L-comodule which is flat as a B-module is the filtered union of L-subcomodules which are finitely generated projective as B-modules.*

5.11. Let (L, c, e) be an R-coalgebroid acting on B. We call a (B, B)-subbimodule M of L a *strict R-subcoalgebroid* if the following conditions are satisfied:

(a) Denote by $i : M \hookrightarrow L$ the inclusion. Then $i \otimes i : M \otimes_B M \to L \otimes_B L$ is injective.
(b) We have $c(M) \subset (i \otimes i)(M \otimes_B M)$.

A strict R-subcoalgebroid carries an induced R-coalgebroid structure. Together with this structure it is a subobject in the category of R-coalgebroids. The converse is in general not true.

Note that (a) holds whenever M and L are flat with respect to both B-module structures.

5.12. Proposition. *Let L be an R-coalgebroid acting on B such that L is flat with respect to both B-module structures. Assume that B is a Prüfer ring. Then L is a filtered union of strict R-subcoalgebroids* (5.11) *which are finitely generated projective with respect to both B-module structures.*

Proof (cf. [3, 4.9]). Via the comultiplication $c : L \to L \otimes_B L$ we consider L itself as an L-comodule. By 5.10 L is filtered union of L-comodules V_i which are projective finitely generated over B. By 2.6 the L-comodule structure on V_i corresponds to an homomorphism of coalgebroids $f_i : V_i^\vee \otimes_R V_i \to L$. Because B is a Prüfer ring and L is flat over B, the image $M_i \subset L_i$ of f_i is a strict R-subcoalgebroid of L which is finitely generated projective over B. The counit e of L induces a linear form λ_i on V_i and for $x \in V_i$ we have $f_i(\lambda_i \otimes x) = x$. Therefore M_i contains V_i and L is the filtered union of the M_i. □

5.13. Let B be an R-algebra which is a Prüfer ring and let L be an R-coalgebroid acting on B such that L is a flat B-module with respect to both B-module structures. Let ω be the forgetful functor from the category of L-comodules over B which are finitely generated projective as B-modules into the category of B-modules.

Theorem. *The canonical homomorphism of R-coalgebroids*

$$u : \underline{\mathrm{CoEnd}}(\omega) \to L$$

is an isomorphism.

Proof. The assumptions of 2.13 hold by 5.5, 5.10, and 5.12. Therefore, we can apply 2.14. □

5.14. We now go back to the general notations of 5.1. Assume that B is commutative. Then the tensor product over B endows the category $\mathcal{C}$ of B-modules with a symmetric monoidal structure and the action of $\mathcal{M}$ is compatible with this monoidal structure in the sense of 2.15. Let $\mathcal{D}$ be a symmetric monoidal category and let φ_1 and φ_2 be two tensor functors $\mathcal{D} \to \mathcal{C}$. Denote by $\underline{\mathrm{Hom}}_B^{\otimes}(\varphi_1, \varphi_2)$ (respectively $\underline{\mathrm{Isom}}_B^{\otimes}(\varphi_1, \varphi_2)$) the presheaf on $(\mathtt{Sch}/\operatorname{Spec}(B))$ which associates to $u : T \to \operatorname{Spec}(B)$ the set of morphisms (respectively isomorphisms) of $\otimes$-functors $u^*\varphi_1 \to u^*\varphi_2$.

Now assume that $\mathcal{D}$ is rigid. Then the functors φ_1 and φ_2 take values in the category of finitely generated projective B-modules (1.4). Therefore the functors $\underline{\mathrm{Hom}}_B^{\otimes}(\omega_1, \omega_2)$ and $\underline{\mathrm{Isom}}_B^{\otimes}(\omega_1, \omega_2)$ are isomorphic [3, 2.7] and representable by affine schemes over B.

On the other hand, applying 2.8 and 2.18, we see that $\underline{\mathrm{CoHom}}_{\mathcal{C}}(\varphi_1,\varphi_2)$ is corepresentable by a commutative B-algebra and the definitions imply (cf. [3, 6.6])

$$\mathrm{Spec}\big(\underline{\mathrm{CoHom}}_{\mathcal{C}}(\varphi_1,\varphi_2)\big)=\underline{\mathrm{Hom}}_B^{\otimes}(\varphi_1,\varphi_2)=\underline{\mathrm{Isom}}_B^{\otimes}(\varphi_1,\varphi_2).$$

5.15. We keep the notations of 5.14. Let $\iota_1,\iota_2 : B\to B\otimes_R B$ be the maps $b\mapsto b\otimes 1$ respectively $b\mapsto 1\otimes b$. Then $\iota_i^*(\varphi_i)$ is a tensor functor from $\mathcal{D}$ into the category of $(B\otimes_R B)$-modules and we have

$$\mathrm{Spec}\big(\underline{\mathrm{CoEnd}}_{\mathcal{M}}(\omega)\big)=\mathrm{Spec}\big(\underline{\mathrm{CoHom}}_{\mathrm{Mod}_{B\otimes_R B}}\big(\iota_1^*(\omega),\iota_2^*(\omega)\big)\big)=\underline{\mathrm{Hom}}^{\otimes}_{B\otimes_R B}\big(\iota_1^*(\omega),\iota_2^*(\omega)\big).$$

The comonoid structure of $\underline{\mathrm{CoEnd}}_{\mathcal{M}}(\omega)$ endows $\mathrm{Spec}(\underline{\mathrm{CoEnd}}_{\mathcal{M}}(\omega))$ with the structure of a monoid scheme. It follows from the definitions (cf. [3, 6.7]) that this corresponds to the composition of morphisms on the right-hand side.

5.16. Now let G be an affine R-groupoid acting on B (where B is commutative) and denote by $s,t : G\to \mathrm{Spec}(B)$ the morphisms source and target. Let $\mathcal{D}$ be the category of representations of G on finitely generated projective B-modules. This is a rigid symmetric monoidal category and we have the canonical forgetful functor $\omega : \mathcal{D}\to\mathcal{C}$ (with the notations of 5.1).

Endowing $G=\mathrm{Spec}(L)$ with the structure of an affine R-groupoid acting on B is equivalent to endowing L with the structure of an *R-Hopfgebroid* acting on B, i.e., with the structure of a Hopf monoid (1.6) in the category $\mathcal{M}$. Further, to give a representation of G on a B-module M is the same as to give M the structure of an L-comodule over B.

5.17. We keep the notations of 5.16 and set

$$\underline{\mathrm{Aut}}_R^{\otimes}(\omega)=\underline{\mathrm{Isom}}^{\otimes}_{B\otimes_R B}\big(\iota_1^*(\omega),\iota_2^*(\omega)\big)$$

with the notations of 5.15. Then $\underline{\mathrm{Aut}}_R^{\otimes}(\omega)$ is an affine R-groupoid acting on B. The target morphism t (respectively source morphism s) is given by composing the projection on $\mathrm{Spec}(B)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(B)$ with pr_1 (respectively pr_2) from $\mathrm{Spec}(B)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(B)$ to $\mathrm{Spec}(B)$.

Theorem. *Assume that B is a Prüfer ring and that t and s are flat morphisms. Then we have a canonical isomorphism of R-groupoids acting on B*

$$G\xrightarrow{\sim}\underline{\mathrm{Aut}}_R^{\otimes}(\omega).$$

Proof. This follows by combining 5.13, 2.18, 5.14, and 5.15. □

5.18. Denote by $\mathcal{M}^f$ (respectively ${}^f\mathcal{M}$, respectively ${}^f\mathcal{M}^f$) the full subcategory of $\mathcal{M}$ which consists of those (B,B)-bimodules which are flat as right B-modules (respectively as left B-modules, respectively as left and a right B-modules).

Lemma. *Let B be a Prüfer ring.*

(1) *Then there exist small inductive limits in these categories.*
(2) *If B is a flat R-algebra the action of ${}^f\mathcal{M}^f$ on $\mathcal{C}$ is coclosed for the subcategory $\mathcal{D}$ of finitely generated projective B-modules.*

Proof. (1) As there exists small direct sums we only have to show that there exist cokernels. Let us first do this for $\mathcal{M}^f$. Let $\varphi : M \to N$ be a homomorphism in $\mathcal{M}^f$. Let C be the cokernel in the category of all (B, B)-bimodules and denote by C^t its right torsion, i.e., C^t consists of those $x \in C$ such that $xb = 0$ for some $0b \in B$. Then C^t is a (B, B)-submodule and C/C^t is a cokernel of φ in $\mathcal{M}^f$ as being flat is equivalent to being torsionfree over a Prüfer ring. Symmetrically, it follows that if φ is a morphism in ${}^f\mathcal{M}$ then $C/{}^tC$ is a cokernel in ${}^f\mathcal{M}$ where ${}^tC$ denotes the left torsion submodule of C. Finally the cokernel in ${}^f\mathcal{M}^f$ is given by $C/(C^t + {}^tC)$.

To prove (2) we have to show by 5.4 that $M^\vee \otimes_R N$ is flat as a left and as a right B-module for all finitely generated projective B-modules M and N. If M and N are free B-modules this follows from the flatness of B over R. In general M and N are direct summands of free modules and this gives (2) as direct summands of flat modules are flat. □

5.19. We keep the assumptions of 5.17 and assume that B is flat over R. Then we can apply 5.13 and 2.18 to ${}^f\mathcal{M}^f$ instead of $\mathcal{M}$ by 5.18 and we get an isomorphism of G with $\operatorname{Spec}(\underline{\operatorname{CoEnd}}_{{}^f\mathcal{M}^f}(\omega))$.

5.20. Corollary. *Let R be a Prüfer ring and let G be an affine flat (and hence faithfully flat) R-group scheme. Denote by $\mathcal{D}$ the category of representations of G on finitely generated projective R-modules and by ω the forgetful functor from $\mathcal{D}$ into the category of R-modules. Then we have a canonical isomorphism of R-group schemes*

$$G \cong \underline{\operatorname{Aut}}_R^{\otimes}(\omega).$$

5.21. In fact, we can associate to every affine R-groupoid G acting on B an affine group scheme over B. The general procedure is as follows.

Let (L, c, e) be an R-coalgebroid acting on B. We can consider the (B, B)-bimodule L as a right $(B^{\mathrm{opp}} \otimes_R B)$-module. As B is commutative, the multiplication $B \otimes_R B \to B$ is a homomorphism of R-algebras and we denote by L^Δ the B-module $L \otimes_{B\otimes_R B} B$. We endow L^Δ with a comultiplication c^Δ defined as the composition

$$L \otimes_{B\otimes_R B} B \xrightarrow{c\otimes \mathrm{id}_B} L \otimes_B L \otimes_{B\otimes_R B} B \xrightarrow{\kappa\otimes \mathrm{id}_B} (L \otimes_{B\otimes_R B} B) \otimes_B (L \otimes_{B\otimes_R B} B),$$

where κ is defined by $x \otimes x' \mapsto x \otimes 1 \otimes x'$ for $x, x' \in L$. Further, we define a counit e^Δ as the composition

$$L \otimes_{B\otimes_R B} B \xrightarrow{e\otimes \mathrm{id}_B} B \otimes_{B\otimes_R B} B \xrightarrow{\sim} B.$$

Then $(L^\Delta, c^\Delta, e^\Delta)$ is a cogebra over B.

If L has the structure of an R-Hopfgebroid, i.e., $G = \mathrm{Spec}(L)$ is an affine R-groupoid acting on B, then multiplication, unit, and antipode of L define on L^{Δ} the structure of a Hopf-algebra over R, i.e., $G^{\Delta} = \mathrm{Spec}(L^{\Delta})$ is an affine group scheme over R. This definition agrees with 4.5.

6. Tannakian lattices over valuations rings of height one

6.1. We fix the following notations. Let R be a valuation ring with field of fractions K. Denote by Γ_R its value group. Every ring homomorphism $\varphi : R \to B$ of valuation rings R and B induces a homomorphism of totally ordered groups $\Gamma_\varphi : \Gamma_R \to \Gamma_B$.

Recall that if Γ is a totally ordered abelian group, a subgroup Γ' of Γ is called *isolated* if the relations $0 < y < x$ and $x \in \Gamma'$ imply $y \in \Gamma'$. The number of isolated subgroups of Γ_R which are distinct from Γ_R is called the *height of* R and denoted by $\mathrm{ht}(R)$. It is equal to the Krull dimension of R.

6.2. Lemma. *Let B be a valuation ring with field of fraction F and let $\varphi : R \to B$ be a homomorphism of rings. Denote by $f : \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ the induced morphism of schemes.*

(1) *The following conditions are equivalent*:
   (i) *f is surjective.*
   (ii) *$\varphi^{-1}(\{0\}) = \{0\}$ and $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_R$.*
   (iii) *φ is injective and via the induced embedding $K \hookrightarrow F$ we have $B \cap K = R$.*
   (iv) *f is faithfully flat and open.*
   (v) *φ and Γ_φ are injective.*
(2) *If the equivalent conditions of* (1) *hold, we have* $\mathrm{ht}(R) \leqslant \mathrm{ht}(B)$. *In particular, R is of finite height if B is of finite height.*
(3) *Let Γ_B be the value group of B and $\Gamma_R \subset \Gamma_B$ the value group of R. Assume that the conditions of* (1) *hold and that B is of finite height. Then the following assertions are equivalent*:
   (i) *The morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ *is bijective* (*and therefore an homeomorphism by* (1)).
   (ii) *We have* $\mathrm{ht}(R) = \mathrm{ht}(B)$.
   (iii) *For every isolated subgroup Δ of Γ_B and for every $x \in \Delta$ there exists a $y \in \Delta \cap \Gamma_R$ such that $y \geqslant x$.*
(4) *Assume that the conditions in* (1) *hold. Then the following assertions are equivalent*:
   (i) *For every $x \in \Gamma_B$ there exists a $y \in \Gamma_R$ such that $y \geqslant x$.*
   (ii) *The homomorphism $B \otimes_R K \to F$ is an isomorphism.*

Proof. Let us prove (1). The implications (iv) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are obvious. Every torsionfree module over a valuation ring is flat [1, Chapitre VI, §3, no. 6, Lemme 1], therefore B is flat over R and we see that (i) and (iii) imply that f is faithfully flat. As the prime ideals of R and B are linearly ordered this implies that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is open. If φ is injective, $B \cap K = R$ also implies $B^\times \cap R = R^\times$ and therefore (iii) implies (v).

Finally, if Γ_φ is injective, Γ_φ sends positive elements in Γ_R to positive element in Γ_B and therefore (v) implies (ii).

Condition (2) then follows from (1) because $\mathrm{ht}(B) = \dim(B) \geqslant \dim(R) = \mathrm{ht}(R)$.

Let us prove (3). The equivalence of (i) and (ii) is clear by (1). Denote by Σ_B (respectively Σ_R) the set of isolated subgroups of Γ_B (respectively Γ_R). These sets are totally ordered by inclusion. The map $\Delta \mapsto \Delta \cap \Gamma_R$ is a surjective map $\Sigma_B \to \Sigma_R$. A right inverse is given by sending $\Delta' \in \Sigma_R$ to the set $I(\Delta')$ of elements $y \in \Gamma_B$ such that there exists an $y' \in \Delta'$ such that $-y' \leqslant y \leqslant y'$. Now assume that there exists an isolated subgroup Δ of Γ_B and an $x \in \Delta$ such that $y < x$ for all $y \in \Delta \cap \Gamma_R$. Let $\Gamma' \in \Sigma_B$ be the isolated subgroup which is the largest among those isolated subgroups of Δ which do not contain x. Then we have $I(\Delta \cap \Gamma_R) \subset \Gamma'$. As $\Gamma' \neq \Delta$ this contradicts (i) and we have proved that (i) implies (iii).

Conversely let $\Delta, \Delta' \in \Sigma_B$ be isolated subgroups such that $\Delta \cap \Gamma_R = \Delta' \cap \Gamma_R$. Let $0 \leqslant x \in \Delta'$. By (iii) there exists a $y \in \Delta' \cap \Gamma_R$ such that $y \geqslant x$. Then we have $y \in \Delta$ and this implies $x \in \Delta$ because Δ is an isolated subgroup. Therefore we see $\Delta' \subset \Delta$. By reversing the roles of Δ and Δ' it follows that $\Delta' = \Delta$ and we have shown that (iii) implies (ii).

Finally, (4) is obvious. □

6.3. Lemma and Definition. *Let F be an extension of K. Then there exists a valuation ring B of F such that $B \cap K = R$ and* $\mathrm{ht}(B) = \mathrm{ht}(R)$ (*and hence $R \subset B$ satisfies all the properties of* 6.2(3) *and* (4), *if R is of finite height*).

We call such a ring B a *height preserving extension of R*.

Proof. Let v be the valuation of K given by R. If F is an algebraic extension every extension of v to F has the same height [1, Chapitre 6, §8, no. 1, Corollaire 1 de Proposition 1]. Therefore, we can assume that F is purely transcendental over K with transcendence basis $(X_i)_{i \in I}$. It follows from loc. cit. §10, no. 1, Proposition 2 that for every finite subset $J \subset I$ there exists a unique extension w of v to $F_J = K((X_i)_{i \in J})$ such that $w(X_i) = 0$ and such that the images of the X_i in the residue class field k_w of w form a transcendental basis of k_w over the residue field of v. Further, the induced inclusion of the value group of v into the value group of w is a bijection. Writing F as the directed inductive limit of the F_J for $J \subset I$ finite we get an extension of v to F with the same value group, in particular the heights are equal. □

6.4. Let F be an extension of K and B a valuation ring of F such that $B \cap K = R$ and such that the heights of R and B are equal and finite. In particular we have $B \otimes_R K = F$ by 6.2. If L is an R-coalgebroid acting on B then L_K (5.3) is a K-coalgebroid acting on $B \otimes_R K = F$. If M is an L-comodule over B then $M \otimes_B F = M \otimes_R K$ is an L_K-comodule over K. This defines a functor from the category $(\mathtt{Proj}_B)^L$ of L-comodules which are finitely generated projective over B into the category $(\mathtt{Vec}_K)^{L_K}$ of L_K-comodules which are finite dimensional vector spaces over K. This induces a tensor functor

$$\Phi : (\mathtt{Proj}_B)^L_K \to (\mathtt{Vec}_K)^{L_K},$$

where $(\mathtt{Proj}_B)^L_K$ denotes the category $(\mathtt{Proj}_B)^L$ with skalars extended to K (3.4).

Proposition. *Assume that L is flat as a left B-module. Then the functor Φ is an equivalence of monoidal categories.*

Proof. We first show that Φ is essentially surjective. Giving an L_K-comodule V over F is equivalent giving an L-comodule V which is a F-vector space. Let V be finite dimensional and choose a B-submodule M of V such that $M \otimes_B F = V$. Then M is finitely generated over B and by 5.7 it is contained in an L-subcomodule N which is finitely generated over B. Further N is projective as a finitely generated submodule of the flat B-module V and we have $N \otimes_B F = V$, i.e., $\Phi(N) = V$.

Now we prove that Φ is fully faithful. Let M and N be two objects in $(\mathtt{Proj}_B)^L$. We have to show that

$$\alpha : \mathrm{Hom}_L(M, N) \otimes_R K \to \mathrm{Hom}_L(M \otimes_B F, N \otimes_B F)$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_L(M, N) \otimes_R K & \xrightarrow{\alpha} & \mathrm{Hom}_L(M \otimes_B F, N \otimes_B F) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_B(M, N) \otimes_R K & \xrightarrow{\sim} & \mathrm{Hom}_B(M \otimes_B F, N \otimes_B F) \\ = \mathrm{Hom}_B(M, N) \otimes_B F & & = \mathrm{Hom}_B(M \otimes_R K, N \otimes_R K), \end{array}$$

where the lower horizontal arrow is bijective because M and N are finitely generated projective. In particular, α is injective. On the other hand, if $f : M \otimes_B F \to N \otimes_B F$ is a B-linear map there exists a $b \in B$ such that $bf(M) \subset N$ because M is finitely generated. By 6.2(3) there exists an $r \in R$ such that $v(r)v(b)$ where v denotes the valuation of B (and its restriction to R). Therefore $rf(M) \subset N$. If f is a homomorphism of L-comodules then this holds for rf as well. This proves the surjectivity of α. $\square$

6.5. If X is any R-scheme then the category $\mathsf{FLF}(X)$ of finite locally free $\mathcal{O}_X$-modules is a rigid symmetric monoidal R-linear category and the canonical functor $\Psi : \mathsf{FLF}(X)_K \to \mathsf{FLF}(X \otimes_R K)$ is a fully faithful tensor functor. Indeed, to show this we can assume that $X = \mathrm{Spec}(A)$ is affine. Denote by S the image of $R \setminus \{0\}$ in A. This is a multiplicative subset and we have $S^{-1}\,\mathrm{Hom}_A(M, N) = \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ if M is an A-module of finite presentation.

6.6. Let L be an R-coalgebroid acting on B and let M and N two L-comodules which are finitely generated projective over B. We can consider $\mathrm{Hom}_L(M, N) \otimes_R K$ and $\mathrm{Hom}_B(M, N)$ as B-submodules of $\mathrm{Hom}_F(M \otimes_B F, N \otimes_B F) = \mathrm{Hom}_B(M, N) \otimes_R K$. Then we have

$$\big(\mathrm{Hom}_L(M, N) \otimes_B F\big) \cap \mathrm{Hom}_B(M, N) = \mathrm{Hom}_L(M, N).$$

6.7. Let $\mathcal{T}$ be a rigid additive symmetric monoidal category such that there is given an isomorphism $R \xrightarrow{\sim} \mathrm{End}_{\mathcal{T}}(1)$ where R is a valuation ring. This endows $\mathcal{T}$ with the structure of an R-linear category. Denote by K the field of fractions of R.

Let X be some R-scheme and ω an R-linear tensor functor from $\mathcal{T}$ with values in the category of quasicoherent $\mathcal{O}_X$-modules. Then ω takes its values in the category of finite locally free $\mathcal{O}_X$-modules (1.4). Further, after skalar extension to K we have a tensor functor ω_K from $\mathcal{T}_K$ with values in the category of finite locally free $\mathcal{O}_{X \otimes_R K}$-modules which is faithful if and only if ω is faithful (6.5).

We consider the following conditions for $\mathcal{T}$:

(TL1) There exists an essentially finite-dimensional R-scheme $X \to \mathrm{Spec}(R)$ (i.e., every local ring of X is finite-dimensional) which is faithfully flat over R and an R-linear tensor functor ω from $\mathcal{T}$ into the category of quasicoherent $\mathcal{O}_X$-modules.

(TL2) $\mathcal{T}_K$ with the induced monoidal structure is a rigid abelian symmetric monoidal category and ω_K is exact.

Note that (TL2) implies that $\mathcal{T}_K$ is a Tannakian category over K.

6.8. Lemma. *Let $\mathcal{T}$ be satisfying* (TL1) *and* (TL2) *and let ω be a functor as in* (TL1) *and* (TL2).

(1) *The functor ω has its values in the category of finite locally free $\mathcal{O}_X$-modules. It is faithful and preserves monomorphisms.*
(2) *For all objects M and N in $\mathcal{T}$ the R-module* $\mathrm{Hom}_{\mathcal{T}}(M, N)$ *is flat.*
(3) *The induced map*

$$\mathrm{Hom}_{\mathcal{T}}(M, N) \otimes_R \mathcal{O}_X \to \mathrm{Hom}_{\mathcal{O}_X}\big(\omega(M), \omega(N)\big) \tag{6.8.1}$$

is injective.

Proof. (1) As $\mathcal{T}$ is rigid, $\omega(M)$ is also rigid for M in $\mathcal{T}$ and therefore finite locally free. By [4, 1.19] the functor ω_K is faithful, hence ω is faithful (3.7). Now let $f : M \to N$ be a monomorphism in $\mathcal{T}$. Then we have a commutative diagram

$$\begin{tikzcd}
\omega(M) \arrow[r, "\omega(f)"] \arrow[d] & \omega(N) \arrow[d] \\
\omega(M) \otimes_R K \arrow[r, "\omega_K(f)"] & \omega(N) \otimes_R K.
\end{tikzcd}$$

As f is a monomorphism, its image in $\mathcal{T}_K$ is also a monomorphism (3.6) and hence the lower horizontal arrow is injective because ω_K is exact. Further the vertical arrows are injective as $\omega(M)$ and $\omega(N)$ are torsionfree over R. This implies that $\omega(f)$ is injective.

(2) As ω is faithful, $\mathrm{Hom}_{\mathcal{T}}(M,N)$ is an R-submodule of $H = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\omega(M), \omega(N)))$ which is flat over R because X is flat over R. Therefore $\mathrm{Hom}_{\mathcal{T}}(M,N)$ is also flat (5.9).

(3) The proof of (3) is the same as in [3, 2.13(ii)] using (1), (2), and that every finitely generated submodule of a flat R-module is free. □

6.9. Definition. Let R be a valuation ring with $\mathrm{ht}(R) \leqslant 1$. A rigid additive symmetric monoidal category $\mathcal{T}$ with a given isomorphism $R \xrightarrow{\sim} \mathrm{End}_{\mathcal{T}}(1)$ is called *quasi-Tannakian lattice over R* if there exists a functor ω as in (TL1) such that $(\mathcal{T}, \omega)$ satisfies (TL2) and the following property:

(TL3) For every height preserving extension B of R and every R-morphism $f : \mathrm{Spec}(B) \to X$ the injection (6.8.1) makes $\mathrm{Hom}_{\mathcal{T}}(M,N) \otimes_R B$ into a direct B-summand of $\mathrm{Hom}_B(f^*\omega(M), f^*\omega(N))$.

A functor as in (TL1), (TL2) and (TL3) is called *fibre functor of $\mathcal{T}$ over X*. A quasi-Tannakian lattice $\mathcal{T}$ is called *Tannakian lattice* if $\mathcal{T}$ is pseudo-abelian.

6.10. For every faithfully flat R-scheme Y and every morphism of R-schemes $f : Y \to X$ the inverse image $f^*\omega : M \mapsto f^*(\omega(M))$ is also a fibre functor.

6.11. Note that (TL3) is equivalent to

(TL3′) For every height preserving extension B of R and every R-morphism $f : \mathrm{Spec}(B) \to X$ the cokernel of the injection $\mathrm{Hom}_{\mathcal{T}}(M,N) \otimes_R B \hookrightarrow \mathrm{Hom}_B(f^*\omega(M), f^*\omega(N))$ is flat.

Indeed, as $f^*\omega(M)$ and $f^*\omega(N)$ are finitely generated projective (and hence free) B-modules, $H := \mathrm{Hom}_B(f^*\omega(M), f^*\omega(N))$ is finitely generated free as well. Therefore any B-submodule H' of H is a direct summand, if and only if the quotient H/H' is flat (which is equivalent to being free as H/H' is finitely generated).

6.12. The pseudo-abelian hull $\mathcal{T}$ of a quasi-Tannakian lattice $\mathcal{T}'$ over R is a Tannakian lattice. Indeed, $\mathcal{T}$ is rigid, symmetrically monoidal and we have $R \xrightarrow{\sim} \mathrm{End}_{\mathcal{T}}(1_{\mathcal{T}})$ by 3.9. Let ω' be a fibre functor of $\mathcal{T}'$ over some faithfully flat R-scheme X. As the category of quasicoherent $\mathcal{O}_X$-modules is abelian this fibre functor factorizes over a functor ω from $\mathcal{T}$ into the category of quasicoherent $\mathcal{O}_X$-modules. As $\mathcal{T}'_K$ is abelian, we have $\mathcal{T}'_K = \mathcal{T}_K$. Hence $(\mathcal{T}, \omega)$ satisfies (TL1) and (TL2), and (TL3) is obvious by the definition of the morphisms in the pseudo-abelian hull.

6.13. If R is a field and $\mathcal{T}$ satisfies (TL1) and (TL2), it also satisfies (TL3) as every height preserving extension B of R is a field extension. Moreover, we have $\mathcal{T} = \mathcal{T}_K$, in particular $\mathcal{T}$ is abelian. Therefore in this case the notions of quasi-Tannakian lattice, of Tannakian lattice, and of Tannakian category in the sense of [3] coincide.

6.14. Proposition. *Let X be an essentially finite-dimensional scheme (i.e., every local ring of X is finite-dimensional) which is faithfully flat over a valuation ring R of height at most one. Then there exists a morphism* $\mathrm{Spec}(B) \to X$ *where B is a height preserving extension of R* (6.3). *If R is noetherian and X is locally of finite type over R, we can assume that B is also noetherian.*

Proof. If R is of height zero, i.e., R is a field, this is trivial. Therefore, assume that R is of height one. As X is faithfully flat over R we can find $x, \eta \in X$ such that x (respectively η) is mapped to the closed (respectively the generic) point of $\mathrm{Spec}(R)$ and such that x is a specialization of η and there exists no other specialization of η which is a generization of x. Let A be the quotient of $\mathcal{O}_{X,x}$ by the prime ideal which is defined by η. Then we have a canonical morphism $\mathrm{Spec}(A) \to X$ and A is a local integral domain of dimension 1. Further, the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is bijective. Therefore the propositions follows from the following lemma.

Lemma. *Let A be a local integral domain of dimension* 1 *with field of fractions F. Then there exists a valuation ring B of F which contains A such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is bijective. If A is noetherian we can assume that B is a discrete valuation ring.*

Proof. Every local subring of F is contained in a valuation ring C of F [1, Chapitre 6, §1, no. 2, Corollaire de Théorème 2] and by localizing C, we see that A is contained in a valuation ring B of height one. We have to show that $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ where $\mathfrak{m}_B$ (respectively $\mathfrak{m}_A$) denotes the maximal ideal of B (respectively A). If this were not the case we would have $\mathfrak{m}_B \cap A = \{0\}$ and this would imply that $A \to B/\mathfrak{m}_B$ is injective which is absurd as A and B have the same field of fractions. The last assertion follows from [5, $\mathbf{0}_I$, 6.5.8]. □

6.15. Corollary. *Let $\mathcal{T}$ be a quasi-Tannakian lattice over a valuation ring R with* $\mathrm{ht}(R) \leqslant 1$. *Then there exists a fibre functor of $\mathcal{T}$ over a faithfully flat R-algebra B which is height preserving extension.*

6.16. Corollary. *Let $\mathcal{T}$ be a quasi-Tannakian lattice over a valuation ring R with* $\mathrm{ht}(R) \leqslant 1$. *Then for all objects M and N in $\mathcal{T}$ the R-module* $\mathrm{Hom}_{\mathcal{T}}(M, N)$ *is finitely generated and free.*

Proof. By 6.8 we know that $H = \mathrm{Hom}_{\mathcal{T}}(M, N)$ is flat over R. Therefore it suffices to show that it is finitely generated. By 6.15 there exists a fibre functor ω over a height preserving extension B of R. By (TL3) we have that $H \otimes_R B$ is a direct summand of the finitely generated B-module $\mathrm{Hom}_B(\omega(M), \omega(N))$. Hence $H \otimes_R B$ itself is finitely generated over B and this implies that H is finitely generated as an R-module because B is faithfully flat over R (6.2(1)). □

6.17. Denote by R a valuation ring with $\mathrm{ht}(R) \leqslant 1$ and by K its field of fractions. Let X be an essentially finite-dimensional scheme which is an *fpqc*-cover of $S = \mathrm{Spec}(R)$ and let G be an R-groupoid acting on X such that $(s, t): G \to X \times_S X$ is affine and faithfully flat. Set $\mathcal{T} = \mathtt{Rep}(X{:}G)$ (4.18) and let ω be the forgetful functor from $\mathcal{T}$ into the category of quasicoherent $\mathcal{O}_X$-modules.

The category $\mathcal{T}$ is R-linear, pseudo-abelian, and carries an obvious symmetric monoidal structure with unit 1 being the trivial representation on $\mathcal{O}_X$. Further, $\mathcal{T}$ is rigid (the dual of a representation $\mathcal{F}$ is given by the contragredient representation $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$).

6.18. Theorem. *$\mathcal{T}$ is a Tannakian lattice over R and ω is a fibre functor.*

Proof. As G is quasicompact and faithfully flat over $X \times_S X$, it acts transitively on X. It follows that the associated stack $\mathcal{G}_{X:G}$ is a gerbe (4.12). By 6.14 there exists an S-morphism $\mathrm{Spec}(B) \to X$ where B is a height preserving extension of R. By 4.21 we can assume that $X = \mathrm{Spec}(B)$. As B is flat over R it follows that the morphisms $s, t : G \to \mathrm{Spec}(B)$ are flat. Further G is affine, say $G = \mathrm{Spec}(L)$.

The unital representation is given by $B \to B \otimes_B L$, $b \mapsto b \otimes 1_L = 1_B \otimes 1_L \cdot b$. Therefore we have $\mathrm{End}_{\mathcal{T}}(1_{\mathcal{T}}) = \{b \in B \mid b \cdot 1_L = 1_L \cdot 1\} =: R'$. This is an R-subalgebra of B. As R is a valuation ring and B is faithfully flat over R, R' is also faithfully flat over R. The groupoid G acts also on $S' = \mathrm{Spec}(R')$ and it is a group scheme over S' (4.4). We claim that R' is a valuation ring. For this denote by K' the field of fractions of R' and by F the field of fractions of B. As every element $y \in F$ is of the form b/r for some $b \in B$ and $r \in R$ (6.2), we see that $K' = \{y \in F \mid y \cdot 1_{L_K} = 1_{L_K} \cdot y\}$, hence $R' = K' \cap B$ which implies that R' is a valuation ring as B is a valuation ring. It follows that B is faithfully flat over R' and therefore G acts transitively on S'. By 4.12 we have $R = R'$ which proves that $\mathrm{End}_{\mathcal{T}}(1_{\mathcal{T}}) = R$.

It remains to prove that (TL2) and (TL3) are satisfied. By 6.4, $\mathcal{T}_K$ is the category of finite dimensional representations on K-vector spaces of G_K and ω_K is the forgetful functor, hence (TL2) is satisfied. To check (TL3) it suffices to consider the case $f = \mathrm{id}_{\mathrm{Spec}(B)}$ by 6.10. Let M and N be two projective finitely generated (hence free) modules over B which are L-comodules. The B-module $H = \mathrm{Hom}_B(M, N)$ is finitely generated, hence the submodule $H' = \mathrm{Hom}_L(M, N)$ is a direct summand iff H/H' is torsion free. By 6.2(3) some B-module has B-torsion if and only if it has R-torsion. But if $f : M \to N$ is a B-linear map such that rf is a homomorphism of L-comodules for some $r \in R \setminus \{0\}$, then f is a homomorphism of L-comodules. $\square$

6.19. Theorem. *Let $\mathcal{T}$ be a quasi-Tannakian lattice over a valuation ring R with* $\mathrm{ht}(R) \leqslant 1$ *and field of fractions K. Then there exists an affine R-groupoid G acting on an R-algebra B such that $\mathcal{T}$ is equivalent to a full sub-tensor category $\mathcal{T}'$ of* $\mathtt{Rep}(G)$ *of representations of G which are finitely generated projective over B and such that $\mathcal{T}'_K = \mathtt{Rep}(G_K)$ where G_K denotes the general fibre of G.*

More precisely, if ω is a fibre functor of $\mathcal{T}$ over a height preserving extension B of R then ω induces a fully faithful functor of $\mathcal{T}$ into the category of representations of the R-groupoid $\underline{\mathrm{CoEnd}}_{f\mathcal{M}^f}(\omega)$ (5.18) *which is after skalar extension to K essentially surjective.*

Further, G is universal with this property in the sense of 2.18(2).

Proof. By 6.15 there exists a fibre functor ω of $\mathcal{T}$ over a height preserving extension B of R. Now use the notations of 5.1 and 5.18. Then $\mathrm{Spec}(\underline{\mathrm{CoEnd}}_{f\mathcal{M}^f}(\omega)) =: G$ is an

R-groupoid acting on B such that source and target $s, t : G \to \operatorname{Spec}(B)$ are flat and ω factorizes through $\mathtt{Rep}(G)$ inducing a functor

$$\omega^G : \mathcal{T} \to \mathtt{Rep}(G).$$

The functor ω_K is a fibre functor of the Tannakian category $\mathcal{T}_K$ over $B \otimes_R K = F$ where F is the field of fractions of B. By [3, 1.12 and 6.7], ω_K induces an equivalence of $\mathcal{T}_K$ with $\mathtt{Rep}(G_K)$ where G_K is the generic fibre of G. This is a K-groupoid acting on F. By 6.4 the canonical functor $\mathtt{Rep}(G)_K \to \mathtt{Rep}(G_K)$ is an equivalence. Therefore ω_G is after skalar extension to K essentially surjective. It remains to show that ω_G is fully faithful. Let M and N be two objects in $\operatorname{Hom}_{\mathcal{T}}(M, N)$. Then we have a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Hom}_{\mathcal{T}}(M,N) & \longrightarrow & \operatorname{Hom}_{\mathrm{Rep}(G)}(\omega(M),\omega(N)) & \longrightarrow & \operatorname{Hom}_B(\omega(M),\omega(N)) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}_{\mathcal{T}}(M,N)\otimes_R K & \xrightarrow{\sim} & \operatorname{Hom}_{\mathrm{Rep}(G_K)}(\omega_K(M),\omega_K(N)) & \to & \operatorname{Hom}_B(\omega(M),\omega(N))\otimes_R K,
\end{array}$$

where all arrows are injective. As the right rectangle is cartesian (6.6) it suffices to show that the composite rectangle is cartesian. For this consider the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Hom}_{\mathcal{T}}(M,N) & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(M,N)\otimes_R B & \longrightarrow & \operatorname{Hom}_B(\omega(M),\omega(N)) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}_{\mathcal{T}}(M,N)\otimes_R K & \longrightarrow & \operatorname{Hom}_{\mathcal{T}}(M,N)\otimes_R F & \longrightarrow & \operatorname{Hom}_B(\omega(M),\omega(N))\otimes_B F.
\end{array}$$

Again all arrows are injective (6.8). As $\operatorname{Hom}_{\mathcal{T}}(M, N)$ is a finitely generated free R-module (6.16), the relation $B \cap K = R$ implies that the left rectangle is cartesian. Further the right rectangle is cartesian because $\operatorname{Hom}_{\mathcal{T}}(M, N) \otimes_R B$ is a direct summand of $\operatorname{Hom}_B(\omega(M), \omega(N))$ by (TL3) and we are done. The last assertion follows from 2.18(2). □

6.20. We keep the hypothesis of 6.19. As $\mathtt{Rep}(G)$ is pseudo-abelian, the fully faithful functor $\mathcal{T} \to \mathtt{Rep}(G)$ factorizes over the pseudo-abelian hull of $\mathcal{T}$ which is a Tannakian lattice (6.12).

6.21. Corollary. *Let $\mathcal{T}$ be a quasi-Tannakian lattice over a valuation ring R with* $\operatorname{ht}(R) \leqslant 1$ *such that there exists a fibre functor ω of $\mathcal{T}$ over R. Then there exists a flat affine group scheme over R and a fully faithful tensor functor $\mathcal{I} : \mathcal{T} \to \mathtt{Rep}(G)$ such that*

(1) *If $\mathcal{F}$ denotes the forgetful functor from $\mathtt{Rep}(G)$ into the category of finitely generated free R-modules, then $\omega = \mathcal{F} \circ \mathcal{I}$.*
(2) *Assume that G' is a second flat affine group scheme over R and that $\mathcal{I}' : \mathcal{T} \to \mathtt{Rep}(G')$ is a tensor functor such that $\omega = \mathcal{F}' \circ \mathcal{I}'$ where $\mathcal{F}'$ is the forgetful functor of $\mathtt{Rep}(G')$.*

Then there exists a unique homomorphism of group schemes $\Phi : G' \to G$ such that $\mathcal{I}'$ is the composition of $\mathcal{I}$ and the functor $\mathtt{Rep}(G) \to \mathtt{Rep}(G')$ induced by Φ.

(3) *If K denotes the field of fractions of R the induced functor after skalar extension to K, $\mathcal{F}_K$, induces an equivalence of categories $\mathcal{T}_K \approx \mathtt{Rep}(G_K)$.*

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References

[1] N. Bourbaki, Commutative Algebra I–VII, Springer-Verlag, Berlin, 1989.
[2] A. Bruguieres, Théorie tannakienne non commutative, Comm. Algebra 22 (14) (1994) 5817–5860.
[3] P. Deligne, Catégories tannakiennes, in: The Grothendieck Festschrift, Collected Articles in Honor of the 60th Birthday of A. Grothendieck, vol. II, in: Progr. Math., vol. 87, 1990, pp. 111–195.
[4] P. Deligne, J.S. Milne, Tannakian categories, in: Hodge Cycles, Motives, and Shimura Varieties, in: Lecture Notes in Math., vol. 900, Springer-Verlag, Berlin, 1982, pp. 101–228.
[5] A. Grothendieck, J.A. Dieudonné, Eléments de Géométrie Algébrique I, Springer-Verlag, Berlin, 1971.
[6] J. Giraud, Cohomologie non abelienne, Grundlehren Math. Wiss., vol. 179, Springer-Verlag, Berlin, 1971.
[7] S. Majid, Braided groups, J. Pure Appl. Algebra 86 (2) (1993) 187–221.
[8] S. Majid, Algebras and Hopf algebras in braided categories, in: J. Bergen, et al. (Eds.), Advances in Hopf Algebras, in: Lecture Notes in Pure and Appl. Math., vol. 158, Dekker, New York, 1994, pp. 55–105.
[9] S. Majid, Reconstruction theorems and rational conformal field theories, Int. J. Mod. Phys. A 6 24 (1991) 4359–4374.
[10] S. Mac Lane, Categories for the Working Mathematician, second ed., Grad. Texts in Math., vol. 5, Springer-Verlag, Berlin, 1998.
[11] R.N. Saavedra, Catégories tannakiennes, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin, 1972.
[12] H. Schubert, Categories, Springer-Verlag, Berlin, 1972.
[13] J.-P. Serre, Groupes de Grothendieck des schemas en groupes reductifs deployes, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 37–52.
[14] S. Shnider, S. Sternberg, Quantum Groups, International Press, 1993.
[15] T. Tannaka, Über den Dualitaetssatz der nichtkommutativen topologischen Gruppen, Tôhoku Math. J. 45 (1938) 1–12.