The Existence of Fiber Functors

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Introduction

This paper contains a short proof of Deligne's theorem on inner characterization of Tannakian categories. The proof was given at Kazhdan's lecture course on topological and conformal field theories at Harvard during the 1990/1991 academic year.

The paper is essentially self-contained. The first section provides preliminaries on linear algebra in monoidal categories, which a reader new to the subject might consider as a series of (mostly quite simple) exercises. The topic of the second section is splitting objects and epimorphisms over appropriate faithfully flat algebra extensions. Then follows the Deligne theorem. Note that the Deligne's original argument [D] is based on a fragment of algebraic geometry in monoidal categories. The argument presented here uses elementary linear algebra, first facts of ring and module theory, in monoidal categories. All the statements (except of the theorem itself) have well known prototypes in conventional linear algebra which makes the proof transparent. Modulo this difference, the argument goes along the same lines as the original one in [D].

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1 Preliminaries

1.1. Monoidal categories Let $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, \alpha, l, r, 1)$, where $\mathcal{C}$ is a category, $\otimes$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, α is a functor isomorphism $\otimes \circ (Id_{\mathcal{C}} \times \otimes) \to \otimes \circ (\otimes \times Id_{\mathcal{C}})$ (*associativity constraint*), and

$$l : Id_{\mathcal{C}} \to 1 \otimes Id_{\mathcal{C}}, \quad r : Id_{\mathcal{C}} \to Id_{\mathcal{C}} \otimes 1$$

are functor isomorphisms.

I. M. Gelfand et al. (eds.), *The Gelfand Mathematical Seminars, 1996-1999*

A morphism from $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, \alpha, \beta, l, r, 1)$ to $\mathcal{C}'^{\sim} = (\mathcal{C}', \otimes', \alpha', \beta', l', r', 1')$, otherwise called a *monoidal functor*, is a triple (F, ϕ, ϕ_0), where F is a functor $\mathcal{C} \to \mathcal{C}'$, $\phi = (\phi_{X,Y})$ is a functorial isomorphism, $\phi_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$ and $\phi_0 : F1 \to 1'$ an isomorphism such that the diagrams

$$\begin{array}{ccccc}
F(X) \otimes' (F(Y) \otimes' F(Z)) & \xrightarrow{id_{F(X)} \otimes' \phi_{Y,Z}} & F(X) \otimes' F(Y \otimes Z) & \xrightarrow{\phi_{X,Y \otimes Z}} & F(X \otimes (Y \otimes Z)) \\
\alpha' \big\downarrow & & & & \big\downarrow F\alpha \\
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{\phi_{X,Y} \otimes' id_{F(Z)}} & F(X \otimes Y) \otimes' F(Z) & \xrightarrow{\phi_{X \otimes Y, Z}} & F((X \otimes Y) \otimes Z)
\end{array} \tag{1}$$

and

$$\begin{array}{ccc}
F(1) \otimes' F(X) & \xrightarrow{\phi_{1,X}} & F(1 \otimes X) \\
\phi_0 \otimes' id \big\downarrow & & \big\downarrow F(l_X) \\
1' \otimes F(X) & \xrightarrow{l'_{F(X)}} & F(X)
\end{array} \qquad
\begin{array}{ccc}
F(X \otimes 1) & \xleftarrow{\phi_{X,1}} & F(X) \otimes' F(1) \\
F(r_X) \big\downarrow & & \big\downarrow id \otimes' \phi_0 \\
F(X) & \xleftarrow{r'_{F(X)}} & F(X) \otimes' 1'
\end{array}$$

are commutative. The composition of morphisms is defined in an obvious way.

The tuple $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, \alpha, l, r, 1)$ is called a *monoidal category* if the triple $(L^{\otimes} : X \to X \otimes -, \alpha, l)$ is a morphism from $\mathcal{C}^{\sim}$ to $(EndA, \circ, id, Id_A, id, id)$.

1.2. Examples (1) The category Vec_k of finite dimensional k-vector spaces with $\otimes = \otimes_k$.

(2) The category of finite dimensional regular representations (over a field k) of an affine algebraic group.

(3) The category of vector super-spaces.

(4) For a given k-linear category $\mathcal{C}$, the category $End_k\mathcal{C}$ of k-linear functors $\mathcal{C} \to \mathcal{C}$ with the composition of functors as $\otimes$ and the identical functor as the unit object: $\otimes = \circ$, $1 = Id_{\mathcal{C}}$.

1.3. Symmetric monoidal categories Let $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, 1, \alpha, l, r)$ be a monoidal category. Then $\mathcal{C}^{\sim\sigma} := (\mathcal{C}, \otimes \circ S, 1, \alpha^{-1}, r, l)$, where S is the transposition functor

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}, \quad (X, Y) \mapsto (Y, X),$$

is a monoidal category called the *opposite monoidal category to* $\mathcal{C}^{\sim}$.

A *symmetry* of monoidal category $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, 1, \alpha, l, r)$ is a functor isomorphism $\beta : \otimes \to \otimes \circ S$ such that $\beta S \circ \beta = id_{\otimes}$ and $(Id_{\mathcal{C}}, \beta, id_1)$ is a norphism from $\mathcal{C}^{\sim}$ to $\mathcal{C}^{\sim\sigma}$. A monoidal category with a fixed symmetry s called *symmetric*.

A morphism from a symmetric category $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, 1, \alpha, l, r; \beta)$ to ι symmetric category $\mathcal{C}'^{\sim} = (\mathcal{C}', \otimes', 1', \alpha', l', r'; \beta')$ is a morphism $F^{\sim} =$ $F, \phi, \phi_0)$ from $\mathcal{C}^{\sim}$ to $\mathcal{C}'^{\sim}$ compatible with the corresponding symmetries. Γhe latter means that the diagram

$$\begin{array}{ccc} F(X) \otimes' F(Y) & \xrightarrow{\phi_{X,Y}} & F(X \otimes Y) \\ \beta'_{F(X),F(Y)} \big\downarrow & & \big\downarrow F\beta_{X,Y} \\ F(Y) \otimes' F(X) & \xrightarrow{\phi_{Y,X}} & F(Y \otimes X) \end{array}$$

:ommutes for all $X, Y \in ObC$.

..4. Algebras, modules and bimodules in monoidal categories

'ix a monoidal category $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, 1, \alpha, l, r)$. An *algebra* (or *monoid*) in $^{\sim}$ is a pair (R, μ) where $R \in ObC$ and μ is a morphism $R \otimes R \to R$ such hat $\mu \circ (\mu \otimes id_R) \circ \alpha_{R,R,R} = \mu \circ (id_R \otimes \mu)$. The *unit* of an algebra (R, μ) is a norphism $\eta : 1 \to R$ such that $\mu \circ \eta \otimes id_R \circ l_R = id_R = \mu \circ id_R \otimes \eta \circ r_R$. The ınit (if it exists) is unique. We assume that all algebras considered here ıre unital. Algebras in $\mathcal{C}^{\sim}$ form a category which we denote by $Alg\mathcal{C}^{\sim}$.

A left module over an algebra (R, μ) is a pair (M, m), where $M \in ObC$, n is a morphism $R \otimes M \to M$ such that $m \circ id_R \otimes m = m \circ \mu \otimes id_M \circ \alpha_{R,R,M}$ ınd $m \circ \eta \otimes id_M = id_M$. Left modules over $R^{\sim} = (R, \mu)$ form a category $?^{\sim} - mod$.

Similarly, we define the category $mod - R^{\sim}$ of right $R^{\sim}$-modules which are just left modules in the opposite monoidal category). A riple (m, M, m'), where (m, M) and (M, m') are resp. left and right $R^{\sim}$-nodules, is called an $R^{\sim}$-*bimodule* if $m \circ id_R \otimes m' = m' \circ m \otimes id_R \circ \alpha_{R,M,R}$.

Suppose the functor $X \mapsto X \otimes -$ is right exact for any $X \in ObC$. Then here is a well defined functor $\otimes_R : mod - R^{\sim} \times R^{\sim} - mod \to C$ which ssigns to any pair of resp. right and left $R^{\sim}$-modules $(M, m), (\nu, N)$ the okernel of the pair of morphisms $id_M \otimes \nu, m \otimes \nu \circ \alpha_{M,R,N} : M \otimes (R \otimes N) \to$ $M \otimes N$. The functor $\otimes_R$ induces a structure of a monoidal category on he category $R^{\sim} - bimod$ of $R^{\sim}$-bimodules.

Let β be a symmetry of the monoidal category $\mathcal{C}^{\sim}$. An algebra $R^{\sim} =$ $R, \mu)$ is called β-*commutative* (or *commutative* if β is fixed) if $\mu \circ \beta_{R,R} = \mu$. 'he full subcategory of $Alg\mathcal{C}^{\sim}$ formed by β-commutative algebras will be enoted by $Alg_{\beta}\mathcal{C}^{\sim}$.

For any β-commutative algebra $R^{\sim}$, the map $(m, M) \mapsto (m, M, m \circ$ $_{M,R})$ defines a functor, Δ_{β}, identifying the category $R^{\sim} - mod$ of left

$R^{\sim}$-modules with a full subcategory of the category $R^{\sim} - bimod$ of $R^{\sim}$-bimodules.

Suppose the functor $X \mapsto X \otimes -$ is right exact for any $X \in ObC$. Then the functor Δ_β identifies $R^{\sim} - mod$ with a monoidal subcategory of $R^{\sim} - bimod$. And the symmetry β induces a symmetry on $R^{\sim} - mod$.

1.4.1. Tensor algebras For any object V of the monoidal category $C^{\sim}$, the tensor powers of V are defined by $V^{\otimes 0} := 1$, $V^{\otimes 1} := V$, $V^{\otimes n} = V \otimes V^{\otimes n-1}$ for $n \geq 2$. Suppose that there exists a direct sum $\oplus_{n\geq 0} V^{\otimes n}$ and $\otimes$ preserves countable direct sums. Then the *tensor algebra*, $\mathbf{T}(V) = (T(V), \mu_V)$ of the object V is defined as follows: $T(V) = \oplus_{n\geq 0} V^{\otimes n}$, and the multiplication μ_V is given by the isomorphisms $V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes n+m}$ induced by the associativity constraint. The map $V \mapsto \mathbf{T}(V)$ is extended to a functor, $\mathbf{T}$, from the category C to the category $AlgC^{\sim}$ of algebras in $C^{\sim}$. This functor is left adjoint to the functor $AlgC^{\sim} \to C$, $(R, \mu) \mapsto R$, forgetting the multiplication.

1.4.2. Symmetric algebras Fix a symmetry β in a monoidal category $C^{\sim}$. Suppose that the functor $X \otimes -$ is right exact for all $X \in ObC$. Then for any algebra (R, μ) in $C^{\sim}$, the cokernel R_β of the pair of morphisms (multiplications) $\mu, \mu \circ \beta_{R,R} : R \otimes R \to R$ has a uniquely defined algebra structure, μ_β. The correspondence $(R, \mu) \mapsto (R_\beta, \mu_\beta)$ extends in a unique way to *a β-abelinization functor Ab_β* : $AlgC^{\sim} \to Alg_\beta C^{\sim}$ from the category of associative algebras to that of β-commutative algebras. The functor Ab_β is left adjoint to the inclusion $Alg_\beta C^{\sim} \to AlgC^{\sim}$.

Let V be an object of C such that there exists a direct sum $\oplus_{n\geq 0} V^{\otimes n}$, hence the tensor algebra of V. We define the *β-symmetric algebra* $\mathbf{S}_\beta(V)$ *of an object* V as the β-abelinization of the tensor algebra $\mathbf{T}(V)$.

1.5. Fragments of linear algebra

1.5.1. Inner hom Let V, W be objects of the monoidal category $C^{\sim}$. The *inner hom* from V to W (if any) is an object, $\mathfrak{Hom}(V, W)$ representing the functor $C(V \otimes -, W) : C^{op} \to Sets$. If $\mathfrak{Hom}(V, W)$ exists, then there is a canonical morphism

$$ev_{V,W} : V \otimes \mathfrak{Hom}(V, W) \to W \tag{1}$$

which is the image of $id_{\mathfrak{Hom}(V,W)}$ under the functorial isomorphism

$$C(\mathfrak{Hom}(V, W), \mathfrak{Hom}(V, W)) \to C(V \otimes \mathfrak{Hom}(V, W), W)$$

Conversely, any morphism $V \otimes X \to W$ determines a morphism $X \to \mathfrak{Hom}(V, W)$. Thus the canonical isomorphism $r_V : V \otimes 1 \to V$ induces a functorial bijection $C(V, W) \to C(1, \mathfrak{Hom}(V, W))$. In particular it defines a morphism $i_V : 1 \to \mathfrak{Hom}(V, V)$ corresponding to the identical morphism id_V.

The inner hom from an object V to the unit object 1 is denoted by $V^\vee$ and is called the *dual object to* V. The canonical arrow $ev_{V,1} : V \otimes V^\vee \to 1$ is called the *evaluation morphism* and is denoted simply by ev_V.

1.5.2. Finite objects Let V, W be objects of C such that $\mathfrak{Hom}(V, W)$ and $V^\vee$ exist. The canonical morphism $ev_V \otimes id_W : V \otimes (V^\vee \otimes W) \to W$ determines a morphism $\phi_{V,W} : V^\vee \otimes W \to \mathfrak{Hom}(V, W)$.

An object V is called *finite* if the morphism $\phi_{V,V}$ is an isomorphism.

1.5.2.1. Finite objects and monoidal functors Another characterization of finite objects: $V \in ObC$ is finite iff there exists an object $V^\vee$ and morphisms $ev : V \otimes V^\vee \to 1$ and $\gamma : 1 \to V^\vee \otimes V$ such that the compositions of $V \xrightarrow{V\otimes\gamma} V \otimes V^\vee \otimes V \xrightarrow{ev\otimes V} V$ and of $V^\vee \xrightarrow{\gamma\otimes V^\vee} V^\vee \otimes V \otimes V^\vee \xrightarrow{V^\vee\otimes ev} V$ are identical morphisms. The notations here are suggestive: $V^\vee$ is a dual object to V. It follows from this description that any monoidal functor (cf. 1.1) sends finite objects into finite objects and a dual object to a given finite object V to a dual object of the image of V.

Another consequence of this description is the following assertion: an object V is finite iff the morphism $\phi_{V,W} : V^\vee \otimes W \to \mathfrak{Hom}(V, W)$ is an isomorphism for all $W \in C$.

1.5.3. Trace and dimension Let $C^\sim = (C, \otimes, a, 1, ...)$ be a symmetric monoidal category with a symmetry β. Let V be a finite object of $C^\sim$. The *β-trace of* V (or simply *trace of* V if the symmetry β is fixed) tr_β is the composition of the canonical isomorphisms $\mathfrak{Hom}(V, V) \to V^\vee \otimes V$ and $\beta_{V^\vee, V} : V^\vee \otimes V \to V \otimes V^\vee$ and the evaluation morphism $ev_V : V \otimes V^\vee \to 1$.

The *dimension* of a finite object V is the composition of the trace and the canonical morphism $j_V : 1 \to \mathfrak{Hom}(V, V)$: $dim(V) = dim_\beta(V) := tr_\beta \circ j_V$.

1.5.3.1. Morphisms of symmetric monoidal categories and dimension It follows from 1.5.2.1 that any morphisms of symmetric monoidal categories $(F, \phi, \phi_0) : (C, \otimes, a, 1, l, r, \beta) \to (C', \otimes', a', 1', l', r', \beta')$ preserves dimensions, i.e., for any finite object V of C, $dim_\beta(V) = dim_{\beta'}(F(V))$.

1.5.4. Symmetric powers The *symmetric power*, $S^n_\beta(V)$, of an object V is the image of the symmetrization $s = \sum_{\sigma \in S_n} \beta_\sigma(V) : V^{\otimes n} \to V^{\otimes n}$, where $\beta_\sigma(V)$ denotes the automorphism of $V^{\otimes n}$ determined by the permutation σ and the symmetry β.

Suppose that $C^\sim$ is a k-category, where k is a field of zero characteristic. Then $S^n_\beta(V)$ is also the image of $s/n!$. And the dimension of $S^n_\beta(V)$ is given by the usual formula:

$$dim S^n_\beta(V) = tr(s/n!) = dim(V)(dim(V)+1)...(dim(V)+n-1)/n! \quad (1)$$

The symmetric algebra $S_\beta(V)$ (cf. 1.4.2) is naturally isomorphic to $(\oplus_{n\geq 0} S^n_\beta(V), \mu)$.

1.5.5. Exterior powers The *exterior power*, $\wedge^n(V)$ of an object V is the image of the antisymmetrization $a = \sum_{\sigma \in S_n} (-1)^{\epsilon(\sigma)} \beta_\sigma(V) : V^{\otimes n} \to V^{\otimes n}$, where $\beta_\sigma(V)$ is the same as in 1.5.4. In the case of zero characteristic, $\wedge^n(V)$ is also the image of $a/n!$. In this case the dimension of $\wedge^n(V)$ is given by the conventional formula:

$$dim(\wedge^n(V)) = tr(a/n!) = dim(V)(dim(V)-1)...(dim(V)-n+1)/n! \quad (1)$$

2 Splitting objects and morphisms in a monoidal category

Fix a monoidal category $\mathcal{C}^\sim = (\mathcal{C}, \otimes, a, 1)$ and its symmetry β.

2.1. Lemma *Let M be a finite object of $\mathcal{C}^\sim$ such that $dim(M)$ is not a nonpositive integer. Then there exists a faithfully flat algebra B such that the B-module $B \otimes M$ admits a decomposition into a direct sum $B \oplus N$ of B-modules.*

Proof. (a) Consider the functor G which assigns to any algebra $\mathcal{R} = (R, \mu)$ in $C^\sim$ the set of all pairs of R-module morphisms $R \xrightarrow{u} R \otimes M \xrightarrow{v} R$. The functor G is isomorphic to the functor G' which assigns to any algebra R the set of all pairs of morphisms $v' : M \to R$, $u' : M^\wedge \to R$. Since $\mathcal{C}(M, R) \simeq Alg(S_\beta(M), R)$, the functor G' is isomorphic to

$$Alg(S_\beta(M \oplus M^\wedge), R) \simeq Alg(S_\beta(M) \otimes S_\beta(M^\wedge), R)$$

(b) The composition $(u, v) \mapsto v \circ u$ is a functor morphism

$$c(R) : G(R) \to R - mod(R, R) \simeq \mathcal{C}(1, R) \simeq Alg(S_\beta(1), R)$$

Let c' denote the corresponding algebra morphism $S_\beta(1) \to S_\beta(M) \otimes S_\beta(M^\wedge)$.

(c) Let Φ be a subfunctor of the functor G which assigns to any algebra R the subset of $G(R)$ consisting of all pairs (u, v) such that $v \circ u = id_R$. The functor $R \mapsto id_R$ is corepresentable by the algebra 1. Hence Φ is corepresentable by the fiber product B of $1 \xleftarrow{\delta} S_\beta(1) \xrightarrow{c'} S_\beta(M) \otimes S_\beta(M^\wedge)$, where δ is the morphism identifying each homogeneous component of $S_\beta(1)$ with 1. The composition of c' with the embedding $i_1 : 1 \to S_\beta(1)$ identifying 1 with the first component of $S_\beta(1)$ is the composition of the canonical morphism $\gamma : 1 \to M \otimes M^\wedge$ and the embedding $M \otimes M^\wedge \to S_\beta(M) \otimes S_\beta(M^\wedge)$. Therefore B is the biggest among quotients of $S_\beta(M) \otimes S_\beta(M^\wedge)$ which coequalize γ and $1 \to S_\beta(M) \otimes S_\beta(M^\wedge)$. It follows that

$$B \simeq \oplus_{m \in \mathbb{Z}} S^n_\beta(M) \otimes S^{n+m}_\beta(M^\wedge),$$

where the transition arrows are multiplications by γ.

(d) The unit $1 \to B$ splits in the category $\mathcal{C}$.

In fact, since $char(k) = 0$, S^n_β is a direct summand of $\otimes^n$ determined by the projection $s/n!$, $s := \sum_{\sigma \in S_n} \beta_\sigma$. In particular, the pairing between $\otimes^n(M^\wedge)$ and $\otimes^n(M)$ induces a pairing (duality) $ev_{S,n} : S^n_\beta(M^\wedge) \otimes S^n_\beta(M) \to 1$. Set $\tau_0 = id$ and $\tau_n = ev_n/d_n$, where $d_n := d(d+1)...(d+n-1)/n!$ is the dimension of $S^n_\beta(M)$. The morphisms τ_n commute with multiplication by γ, $\gamma_n : S^n_\beta(M^\wedge) \otimes S^n_\beta(M) \to S^{n+1}_\beta(M^\wedge) \otimes S^{n+1}_\beta(M)$, $\tau_{n+1} \circ \gamma_n = \tau_n$. Hence morphisms $\{\tau_n | n \geq 0\}$ define a morphism τ from the direct summand (and a subalgebra) $B' = colim(S^n_\beta(M^\wedge) \otimes S^n_\beta(M))$ of the algebra B to 1. The morphism τ is left inverse to the unit $1 \to B$. $\square$

2.2. Lemma *Let $v : M \to N$ be an epimorphism of finite objects. Then there exists a faithfully flat algebra B such that the morphism $id_B \otimes v : B \otimes M \to B \otimes N$ splits.*

Proof. (a) Suppose that $N = 1$, and let $v^\wedge$ denote the dual to v morphism $1 \to M^\wedge$. Let B denote the fiber product of the pair of arrows $1 \longleftarrow S_\beta(1) \xrightarrow{Sv^\wedge} S_\beta(M^\wedge)$. Clearly $B \simeq colim S^n_\beta(M^\wedge)$, where translation arrows $S^n_\beta(M^\wedge) \to S^{n+1}_\beta(M^\wedge)$ are the multiplication by $v^\wedge$. The canonical morphism $\gamma : 1 \to M^\wedge \otimes M$ defines a B-module morphism $\phi : B = B \otimes 1 \to B \otimes B \otimes M$ which is right inverse to $id_B \otimes v$.

Note that a monomorphism $V \to W$ of finite objects induces a filtration on $\otimes^n(W)$. Since the restriction of the functor $\otimes^n$ to the subcategory of finite modules is exact, the associated graded module is $\otimes^n(V \otimes W/V)$. Applying the projection $s = (\sum_{\sigma \in S_n} \sigma)/n! : \otimes^n \to S^n_\beta$, we obtain the corresponding filtration of $S^n_\beta(W)$ with the quotients $S^i_\beta(V) \otimes S^{n-i}_\beta(W/V)$, $0 \le i \le n$. Applying this to the monomorphism $v^\wedge : 1 \longrightarrow M^\wedge$, we obtain the filtration

$$0 \longrightarrow 1 = S^0_\beta(M^\wedge) \xrightarrow{v^\wedge} M^\wedge = S^1_\beta(M^\wedge) \xrightarrow{v^\wedge} \ldots \xrightarrow{v^\wedge} S^n_\beta((M^\wedge)$$

with quotients $S^i_\beta(M^\wedge/1)$. This shows that the algebra B is faithfully flat.

(b) Let now $v : M \longrightarrow N$ be an arbitrary epimorphism of finite objects. Since N is finite, in particular the functor $\mathfrak{Hom}(-, N)$ is exact, the morphism $v' = \mathfrak{Hom}(v, id_N)\mathfrak{Hom}(M, N) \longrightarrow \mathfrak{Hom}(N, N)$ is an epimorphism. Let M' be the fiber product of the morphism v' and the canonical morphism $1 \longrightarrow \mathfrak{Hom}(N, N)$. Since the category C is abelian, the projection $\pi : M' \longrightarrow 1$ is an epimorphism. Given a commutative algebra B, the morphism $id_B \otimes v$ splits iff the morphism $id_B \otimes \pi$ splits. Thus the assertion follows from the part (a) of the argument. □

3 Deligne's theorem

Given a k-linear symmetric monoidal category $C^\sim$, a fiber functor over a commutative k-algebra R is a k-linear morphism (F, ϕ, ϕ_0) from $C^\sim$ to the symmetric monoidal category of R-modules over a commutative k-algebra R such that the functor F is exact and faithful. The monoidal category $C^\sim$ is called *Tannakian* if it admits a fiber functor over some nonzero commutative ring.

3.1. Theorem *Let $C^\sim = (C, \otimes, 1, \beta)$ be a rigid symmetric monoidal category over a field k of zero characteristic such that $C(1, 1) = k$. Then the following conditions are equivalent:*

(a) The monoidal category $C^\sim$ is Tannakian.

(b) The dimension of each object of $C^\sim$ is a nonnegative integer.

(b′) The dimension of each nonzero object of $C^\sim$ is a positive integer.

(c) For each object X of C, there exists an integer n such that $\wedge^n(X) = 0$.

Proof. $(a) \Rightarrow (b)$. By 1.5.2.1 and 1.5.3.1 fiber functors send finite objects to finite objects and preserve dimensions. Finite objects in the category of vector spaces are finite dimensional vector spaces and the dimension in the sense of 1.5.3 coincides in Vec_K with the usual dimension over K, hence the assertion.

$(b) \Leftrightarrow (c)$. This follows from the formula 1.5.5(1).

$(b) \Rightarrow (b')$. Suppose that V is a nonzero object of $\mathcal{C}$ such that $dim(V) = 0$. Since $V \neq 0$, the canonical morphism $\eta_V : 1 \to V^\vee \otimes V$ is nonzero, hence injective. Let W be a cokernel of η_V. We have: $dim(W) = dim(V^\vee \otimes V) - dim(1) = dim(V^\vee) dim(V) - 1 = -1$ which contradicts the assumption that $dim(W) \geq 0$.

The implication $(b') \Rightarrow (b)$ is obvious.

$(b')\&(c) \Rightarrow (a)$ (i) Any rigid abelian (symmetric) monoidal category $C^\sim$ is canonically embedded into an abelian monoidal symmetric category $C'^\sim = (C', \otimes', a', 1, ...)$ such that C' has colimits and C is the full subcategory of C' generated by all finite objects of $C'^\sim$. Namely the category C' is the category $IndC$ of *ind-objects* of C (SGA4, I.8.2). Recall that objects of $IndC$ are filtered inductive systems (V_i) in C and morphisms from (V_i) to (W_j) are given by $C'((V_i), (W_j)) = lim_i colim_j C(V_i, W_j)$. The bifunctor $\otimes$ induces a bifunctor $\otimes'$ which is also exact. The associativity constraint a and the symmetry constraint β induce resp. an associativity and symmetry constraints in $C'^\sim$.

(ii) Fix a finite object X of the monoidal category $\mathcal{C}^\sim$. And let $dim(X) = d$ be an positive integer. By 2.1, there exists a faithfully flat algebra R in $C'^\sim$ such that $R \otimes X \simeq (d) X \oplus N$ for some R-module N. For any two objects, V and W, we have: $\Lambda^n(V \oplus W) \simeq \oplus_{i+j=n} \Lambda^i(V) \otimes \Lambda^j(W)$. In particular, N is a direct summand in $\Lambda^{d+1}(R \otimes X) = R \otimes \Lambda^{d+1}(X) = 0$, hence $N = 0$.

(iii) It follows from (i) that there exists a commutative faithfully flat algebra B in $\mathcal{C}'^\sim$ such that $B \otimes X \simeq (dimX) B$ for any finite object X of $\mathcal{C}^\sim$. By 2.2, there exists a faithfully flat B-algebra R in $\mathcal{C}'^\sim$ such that for any short exact sequence $\mathcal{E}$ in $B - mod$, the sequence $R \otimes_B \mathcal{E}$ splits. It follows that the functor $X \mapsto C'(1, R \otimes X)$ is a fiber functor over the commutative k-algebra $C'(1, R)$. □

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