TANNAKIAN DUALITY OVER DEDEKIND RINGS AND APPLICATIONS

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Dedicated to Hélène Esnault, with admiration and affection

ABSTRACT. We establish a duality between flat affine group schemes and rigid tensor categories equipped with a neutral fiber functor (called Tannakian lattice), both defined over a Dedekind ring. We use this duality and the known Tannakian duality due to Saavedra to study morphisms between flat affine group schemes. Next, we apply our new duality to the category of stratified sheaves on a smooth scheme over a Dedekind ring R to define the relative differential fundamental group scheme of the given scheme and compare the fibers of this group scheme with the fundamental group scheme of the fibers. When R is a complete DVR of equal characteristic we show that this category is Tannakian in the sense of Saavedra.

INTRODUCTION

Tannakian duality for group scheme over a field was studied by Saavedra [21]. The duality in the neutral case, as shown by Saavedra, is a dictionary between k-linear abelian rigid tensor categories equipped with a fiber functor to the category of k-vector spaces and affine group schemes over k. The duality consists of two parts:

- The reconstruction theorem which recovers a group scheme from a neutral Tannakian category $(\mathfrak{T}, \omega : \mathfrak{T} \longrightarrow \mathsf{Vect}(k))$, as the group of automorphisms of ω preserving the tensor product, the Tannakian group of (\mathfrak{T}, ω) .
- The presentation (or description) theorem which claims the equivalence between the original category T and the representation category of the Tannakian group of T.

Saavedra also extended this result to the non-neutral case - when the fiber functor goes to a more general category of coherent sheaves over a k-scheme. A complete proof of this theorem was given by Deligne in [6].

An important application of Tannakian duality is to define various fundamental group schemes. Let X be a scheme over a field k. There are certain abelian tensor categories associated to X. For example, if k is perfect, X is reduced and connected, M. Nori introduced the category of essentially finite bundles; if X is smooth and k has characteristic zero, one has the category of flat connections on X; if k has positive characteristic, one has the category of stratified bundles (i.e. \mathcal{O}_X -coherent modules equipped with the action of the sheaf $\mathscr{D}_{X/k}$ of algebras of differential operators on X). Given a k-rational point x of X, the functor taking fibers at x makes the above categories Tannakian category and Tannakian duality yields the corresponding affine

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group scheme, which is usually called the fundamental group scheme of X. Tannakian duality is also used as an alternative approach to the Picard-Vessiot theory of linear differential equations.

Let now $f : X \longrightarrow S :=$ SpecR be a smooth morphism, where R is a Dedekind ring. We are interested in the category of modules over $\mathscr{D}(X/S)$, which are coherent as \mathcal{O}_X -modules, where $\mathscr{D}(X/S)$ denotes the sheaf of algebras of differential operators on X/S. Such a sheaf will be called stratified sheaf over X/S. The category str(X/S) of stratified sheaves on X/S is an abelian tensor category, in which an object is rigid if it is locally free as an \mathcal{O}_X -module. Assume that f admits a section $\xi : S \longrightarrow X$. Then the functor ξ^* provides us a fiber functor for str(X/S). It is then natural to ask, if there exists a generalization of Tannakian duality to this case.

In fact, Tannakian duality over Dedekind rings has already been considered by Saavedra in [21, II.2]. Saavedra gave a condition for an abelian category equipped with an exact faithful functor to the module category (over a given Noetherian ring) to be equivalent to the comodule category over the coalgebra reconstructed from this functor. This duality is then developed to a Tannakian duality for flat affine group schemes over a Dedekind ring.

In many examples, the involving category may not satisfy all the properties in Saavedra's definition. One therefore wants to reconstruct an affine group scheme from a smaller part of its representation category, for instance, from rigid representations (i.e. representations in finite projective modules). There are at least two approaches to this problem, one by Wedhorn [29], and the other by Bruguiéres [3] following an idea of M. Nori, the latter one has been used by dos Santos [23] to define the Galois differential groups of a relative stratified bundle. Wedhorn's approach is similar to Saavedra's approach. He introduced the notion of Tannakian lattice and reconstructed a group scheme from such a lattice. However Wedhorn has not established a full duality; he did not show that a Tannakian lattice is equivalent to the category of finite projective representations of a flat affine group scheme.

In Section 1 we first recall the definition of the representation category of a flat affine group scheme over a Dedekind ring. A special property of such a category, as noticed by Serre [25], is that any object can be represented as a quotient of a projective one (over the base ring). Saavedra uses this feature to characterize these representation categories (Theorem 1.2.2). A self-contained proof of this theorem will be given in the Appendix.

In Section 2 we propose a new definition of neutral Tannakian lattices. Saavedra's method is used to establish a full duality between neutral Tannakian lattices and flat affine group schemes over a Dedekind ring (Theorem 2.3.2). As a corollary we show that a Tannakian lattice is contained in a unique (up to equivalence) abelian envelope (Proposition 2.4.4). We also construct a torsor of isomorphisms between two fiber functors.

The Tannakian duality is used in Section 3 to study properties of homomorphisms of flat coalgebras over a Dedekind ring. We first introduce the notions of special coalgebra homomorphisms and special subcoalgebras (Definition 3.1.1). For coalgebras over a field, one has the local finiteness property: each coalgebra is the union of its finite dimensional subcoalgebras. The situation becomes more complicated for flat coalgebras over a Dedekind ring. A flat coalgebra over a Dedekind ring is still the union of its finite subcoalgebras, but one cannot choose the subcoalgebras to be at the same time finite and saturated as submodules. We introduced the notion of special locally finite coalgebras over a Dedekind ring, which are those representable as union of finite saturated subcoalgebras. We show that the coordinate ring of a flat group scheme, the generic fiber of which is reduced and connected, is specially locally finite (Proposition 3.1.7). In the second part of this section we provide conditions for a coalgebra homomorphism to be (specially) injective or surjective (Propostions 3.2.1, 3.2.5).

In Section 4 the results on homomorphisms of flat coalgebras are used to study homomorphisms of flat group schemes: to characterize closed immersions and the faithful flatness. In the last part of Section 4, we give a criterion for the exactness of sequences of homomorphisms of flat affine group schemes over Dedekind rings in terms of Tannakian duality. Such a criterion for group schemes over a field has proved to be a useful tool, see [9, 11, 10, 15]. Some applications of the results in this section will be presented in a subsequent paper [8].

In Section 5 we apply the Tannakian duality to the category str(X/R) of stratified sheaves over a smooth scheme X/R, where R is a Dedekind ring. We show that the subcategory $str^{o}(X/R)$ of relative stratified bundles is a Tannakian lattice (assuming the existence of an R-point of X). We then compare the fibers of the relative fundamental group and the fundamental group of the fibers. When R is a complete local discrete valuation ring of equal characteristics, we show that the category of all stratified sheaves str(X/R) is a Tannakian category over R.

In the Appendix, for the sake of the reader, we recall Saavedra's proof of Tannakian duality for flat affine group schemes over a Dedekind ring.

1. PRELIMINARIES

In this article R will denote a *Dedekind ring*. The fraction field of R is denoted by K, a residue field will be denoted by k. The category of R-modules is denoted by Mod(R), its full subcategory of finite modules is denoted by $Mod_f(R)$ and the full subcategory of finite projective modules is denoted by $Mod^o(R)$. We shall intensively use the facts that over a Dedekind ring, torsion free modules are flat and finitely generated flat modules are projective. The tensor product of R-modules, when not explicitly indicated, is understood as the tensor product over R.

1.1. Flat affine group schemes. Let G be a flat affine group scheme over R. The coordinate ring of G, R[G], is usually denoted by L for short. Thus L is an R-flat (commutative) Hopf algebra.

1.1.1. By definition, a G-module (or a G-representation) is the same as a *right* L-comodule, i.e., an R-module M equipped with a coaction of L:

$$\rho: \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathbb{R}} \mathsf{L}; \quad (\rho \otimes \mathsf{id})\rho = (\mathsf{id} \otimes \Delta)\rho, \quad (\mathsf{id} \otimes \varepsilon)\rho = \mathsf{id},$$

where $\Delta : L \longrightarrow L \otimes L$ denotes the coproduct and $\varepsilon : L \longrightarrow R$ denotes the counit of L.

The flatness of L implies that the category Comod(L) of (right) R-modules equipped with a coaction of L is an R- linear abelian category. We call an L-comodule M *finite* if it is finite as an R-module and *finite projective* if it is also projective over R. The full subcategory of finite L-comodules is denoted by $Comod_f(L)$ and the full subcategory of finite projective L-comodules is denoted by $Comod^o(L)$. Each L-comodule is the union of its finite subcomodules (see Appendix).

1.1.2. Comod(L) is a tensor category with respect to the tensor product over R. The unit object is R equipped with the trivial coaction of L: $R \longrightarrow R \otimes L = L$, $1 \mapsto 1 \otimes 1$. The subcategory Comod^o(L) is rigid, i.e., each objects possesses a dual.

1.1.3. More general, a comodule J is said to be trivial if the coaction maps any element m to $m \otimes 1 \in J \otimes L$. A finite trivial comodule is thus a quotient of the (trivial) comodule R^n . By taking duality we see that a rigid trivial comodule is a subcomodule of R^n . For a comodule V, the maximal trivial subcomodule V^{triv} of consists of elements ν such that $\rho(\nu) = \nu \otimes 1$.

1.1.4. Let V be a finite projective L-comodule. The coaction $\rho: V \longrightarrow V \otimes L$ induces a map

$$\mathsf{Cf}: V^{\vee} \otimes V \longrightarrow \mathsf{L}, \quad \phi \otimes \mathfrak{m} \mapsto \sum \phi(\mathfrak{m}_i)\mathfrak{m}'_i, \quad \phi \in V^{\vee}, \mathfrak{m} \in \mathsf{V}, \Delta(\mathfrak{m}) = \sum_i \mathfrak{m}_i \otimes \mathfrak{m}'_i.$$

This map can be considered either as a homomorphism of L-comodules, where L coacts on itself by the coproduct and coacts on $V^{\vee} \otimes V$ by the action on the second tensor component, or as a homomorphism of coalgebras. The image of this map, denoted by Cf(V), is called the coefficient space of V. Since L is flat, it is a subcoalgebra of L, i.e., $\Delta(Cf(V)) \subset Cf(V) \otimes Cf(V)$ (cf. Lemma 3.1.4 (i)). The coaction of L on V factors through the coaction of Cf(V) on V.

We note that our requirement here for a subcoalgebra is weaker than in some other literatures, e.g. [14], where a subcoalgebra is a special subcoalgebra in our sense (see. 3.1.1 for definition).

1.1.5. Following dos Santos, we call a subcomodule U of V *special* if U is saturated in V, i.e., V/U is R-flat. For instance, the coaction $V \rightarrow V \otimes L$ can be considered as a comodule map (where L coacts on the target by the coaction on itself). It splits as an R-module map by means of the counit $\varepsilon : L \rightarrow R$ of L. Hence, if V is R-flat, V is a special submodule of $V \otimes L$.

If V is finite projective, then Cf(V) contains both Cf(U) and Cf(V/U), [23, Lem. 9]. In deed, this inclusion property is a local property, so we can assume that R is a DVR, then V, U and V/U are all free over R. Hence we can choose a basis $(e_1, e_2, ..., e_n)$ of V such that the first m elements form a basis of U and the cosets of the other elements form a basis of V/U. The coaction of L on V with respect to this basis is given in terms of a multiplicative matrix (a_i^j) :

$$\rho(e_i) = \sum_{j=1}^n e_j \otimes a_i^j, \quad i = 1, \dots, n.$$

Notice that the elements a_i^j are uniquely determined by the basis elemnents e_i 's. Now the assumption that U is a subcomodule implies that $a_i^j = 0$ for i = 1, ..., m and j = m + 1, ..., n. It follows that the coaction of L on V/U with respect to the basis $(\overline{e_{m+1}}, ..., \overline{e_n})$ is given by

$$\rho(\overline{e_k}) = \sum_{l=m+1}^{n} \overline{e_l} \otimes a_k^l, \quad k = m+1, \dots, n.$$

For example, for an R-flat comodule V, the maximal trivial subcomodule V^L of V is special. Indeed, if we have $0 \neq v = au$, $a \in R$, $u \in V$ then from the equality $\rho(au) = au \otimes 1$ and the flatness of $V \otimes L$, we conclude that $\rho(u) = u \otimes 1$. Thus V/V^L is torsion free, hence flat. This implies that for any two finite projective comodules V, W, the inclusion

$$\operatorname{Hom}^{\mathsf{L}}(V, W) \longrightarrow \operatorname{Hom}_{\mathsf{R}}(V, W)$$

is saturated. Indeed, we have

$$\operatorname{Hom}^{\operatorname{L}}(V,W) \cong \operatorname{Hom}^{\operatorname{L}}(\mathsf{R},W\otimes V^{\vee}) = (W\otimes V^{\vee})^{\operatorname{L}} \subset W\otimes V^{\vee}.$$

1.1.6. For a finite comodule V, we denote by $\langle V \rangle$ the full subcategory generated by V, i.e., consisting of subquotients of finite direct sums of V. For a finite projective comodule V, denote by $\langle V \rangle_s$ the full subcategory specially generated by V, i.e. consisting of special subquotients (quotients of special subobjects or special subobjects of quotients objects) of direct sums of copies of V.

Lemma 1.1.7. Let V be a finite projective comodule over a flat R-coalgebra L. Then the category $\langle V \rangle_s$ is equivalent with Comod^o(Cf(V)) (by means of the obvious functor which is the identity functor on the underlying R-modules).

Proof. Consider the restriction functor $Comod^{o}(Cf(V)) \longrightarrow Comod^{o}(L)$. The condition for a map $\varphi : M \longrightarrow N$ to be an L-comodule map reads as follows: the map $\rho_N \varphi - (\varphi \otimes id)\rho_M : M \longrightarrow N \otimes L$ is the zero map (i.e. the outer square in the diagram below is commutative).

$$\begin{array}{c} M \xrightarrow{\rho_{M}} M \otimes Cf(V) \longrightarrow M \otimes L \\ \varphi \downarrow & \varphi \otimes id \downarrow & \downarrow \varphi \otimes id \\ N \xrightarrow{\rho_{N}} N \otimes Cf(V) \longrightarrow N \otimes L. \end{array}$$

If N is flat over R, the horizontal map $N \otimes Cf(V) \longrightarrow N \otimes L$ in the above diagram is injective. Hence the condition for φ to be an L-comodule map is the same as the condition for φ to be a Cf(V)-comodule map (which amounts to the left square to commute). Thus the restriction functor Comod^o(Cf(V)) \longrightarrow Comod^o(L) is fully faithful.

On the other hand, according to 1.1.5, if $W \in \langle V \rangle_s$ then $Cf(W) \subset Cf(V)$, hence W is a subcomodule of Cf(V). Thus, it remains to show that any finite projective comodule over Cf(V) is a special subquotient of a finite direct sum of copies of V.

This claim holds for Cf(V) itself, as, by definition, Cf(V) is a quotient of $V^{\vee} \otimes V$, where Cf(V) coacts on $V^{\vee} \otimes V$ by the coaction on the second tensor component. Further, if M is a finite projective Cf(V)-comodule then M is a special subcomodule of $M \otimes Cf(V)$ (cf. 1.1.5), i.e. it is a special subquotient of a direct sum of copies of V. The proof is complete.

1.1.8. *Warning*. The functor $Comod_f(Cf(V)) \longrightarrow Comod(L)$ is faithful and exact but generally not full, see Section 3 below. It is not clear to us how to specify the image of this functor.

1.1.9. For a comodule M, its (R-) torsion part M^{tor} is also a subcomodule. The quotient M/M^{tor} is R-torsion free, hence flat, hence R-projective if it is finite over R.

1.1.10. L-comodules are locally finite, i.e. they are union of their subcomodules of finite type. In fact, for each finite set S of a comodule M, there exists a subcomodule N, finitely generated over R, which contains S. It follows that L itself is the union of its subcoalgebras, which are finite over R, c.f. [25, Cor. 2]. See also Appendix A.1.4.

1.1.11. Let K be the fractions field of R. Denote $L_K := L \otimes_R K$. Then L_K is a Hopf algebra over the field K. If V is a comodule over L, then $V_K := V \otimes_R K$ is a comodule over L_K . For any two finite projective comodules V, W, the natural map

 $\operatorname{Hom}^{L}(V, W) \otimes_{R} K \longrightarrow \operatorname{Hom}^{L_{K}}(V_{K}, W_{K})$

is an isomorphism. Indeed, if $f \in \text{Hom}^{L_{K}}(V_{K}, W_{K})$, then there exists $0 \neq a \in R$ such that $af: V \longrightarrow W$. But then we have $f = af \otimes a^{-1}$.

Conversely, let X be a finite dimensional comodule over L_K . Since $X \otimes_K L_K \cong X \otimes_R L$, X is an L-comodule. Let (e_1, \ldots, e_n) be a basis of X. Then there exists a finite L-comodule $V \subset X$, which contains (e_1, \ldots, e_n) . Now V is finite hence projective over R. As V contains a basis of X, we have $V_K \cong X$. Note that V is not unique, but all such V has the same rank, which is the dimension of X over K.

1.2. Tannakian duality for abelian tensor categories.

Definition 1.2.1 (Subcategory of definition, cf. [21, II.2.2]). Let \mathcal{C} be an R-linear abelian category, and $\omega : \mathcal{C} \longrightarrow \mathsf{Mod}_f(\mathsf{R})$ be an R-linear exact faithful functor. Suppose that \mathcal{C}^o is a full subcategory of \mathcal{C} such that:

- (i) for any object $X \in C^{o}$, $\omega(X)$ is a finitely generated projective R-module;
- (ii) every object of C is a quotient of an object of C° .

Then \mathbb{C}° is called a *subcategory of definition* of \mathbb{C} with respect to ω .

This definition is motivated by the following fact, due to Serre (see [25, Prop.3]). For any finite L-comodule E there exists a short exact sequence of L-comodules

$$0 \longrightarrow \mathsf{F}' \longrightarrow \mathsf{F} \longrightarrow \mathsf{E} \longrightarrow 0,$$

in which F' and F are R-finite projective. Thus, the subcategory $Comod^{o}(L)$ of R-finite projective L-comodules is a subcategory of definition in $Comod_{f}(L)$.

Theorem 1.2.2 (cf. [21, Thm. II.2.3.5, II.2.6.1]). Let R be a Noetherian ring and let C be an R-linear abelian category. Assume that there exist an R- linear exact faithful functor $\omega : \mathbb{C} \longrightarrow Mod_f(R)$ and a subcategory of definition \mathbb{C}^o with respect to ω . Then ω factors through an equivalence $\mathbb{C} \simeq Comod_f(L)$ and the forgetful functor, for some flat R-coalgebra L.

Although this theorem is formulated for any Noetherian ring, We don't know any examples of comodule category satisfying the conditions of Definition 1.2.1 when R is a ring of dimension larger than 1. A self-contained proof of this theorem will be given in the Appendix.

1.2.3. The coalgebra L in the theorem above can be determined from the fiber functor ω as follows. We claim that there is a natural isomorphism

(1)
$$\operatorname{Nat}(\omega, \omega \otimes M) \simeq \operatorname{Hom}_{R}(L, M),$$

for any R-module M, see Appendix A.1.10. If $\mathcal{C} = \text{Comod}_f(L)$ and ω is the forgetful functor from \mathcal{C} to Mod(R), then the above isomorphism implies that $\text{Coend}(\omega) \simeq L$. In particular, a flat coalgebra over R can be *reconstructed* from the category of its comodules. The isomorphism in (1) implies that, for any R-algebra A,

$$\operatorname{Nat}_{A}(\omega \otimes A, \omega \otimes A) \simeq \operatorname{Hom}_{R}(L, A).$$

If C is a tensor category and ω is a (strict) tensor functor, then L is a bialgebra and we have an isomorphism

$$\operatorname{Nat}_{A}^{\otimes}(\omega \otimes A, \omega \otimes A) \simeq \operatorname{Hom}_{R-Alg}(L, A),$$

where Nat^{\otimes} denotes the set of natural transformations that are compatible with the tensor product.

1.2.4. Assume that C is an R-linear abelian tensor category. The reader is referred to [5, Sect. 1] for the notion of dual objects. An object is called rigid if it possesses a dual. Notice that the image of a rigid object under a tensor functor to $Mod_f(R)$ is a finite projective module. Denote by C^o the full subcategory of C consisting of rigid objects. We say that C is *dominated* by C^o if each object of C is a quotient of a rigid object.

Definition 1.2.5. A (neutral) Tannakian category over a Dedekind ring R is an R-linear abelian tensor category \mathcal{C} , dominated by \mathcal{C}° , together with an exact faithful tensor functor $\omega : \mathcal{C} \longrightarrow Mod(R)$. In this case, \mathcal{C}° is a subcategory of definition in \mathcal{C} .

Let ω° denote the restriction of ω to \mathcal{C}° . Then we have, for any R-algebra A,

 $\mathsf{Nat}^\otimes_A(\omega\otimes A,\omega\otimes A)\simeq \mathsf{Nat}^\otimes_A(\omega^o\otimes A,\omega^o\otimes A)\simeq \mathsf{Aut}^\otimes_A(\omega^o\otimes A,\omega^o\otimes A).$

For the first isomorphism see the proof of Lemma A.1.6, for the second isomorphism see [5, Prop. 1.13].

Theorem 1.2.6 ([21, Thm. II.4.1.1]). Let (\mathcal{C}, ω) be a neutral Tannakian category over a Dedekind ring R. Then the group functor $A \mapsto \operatorname{Aut}_{A}^{\otimes}(\omega \otimes A)$ is representable by a flat group scheme G and ω factors through an equivalence between \mathcal{C} and $\operatorname{Rep}_{f}(G)$.

1.3. **Scalar extension.** In [29], Wedhorn proposes to establish a duality for rigid tensor category over a Dedekind ring. In fact, in many examples, it is easy to specify a rigid tensor category, but it is very difficult to determine a Tannakian category containing the given monoidal category as a subcategory of definition. The natural problem is to characterize intrinsically the subcategory of rigid objects in a Tannakian category (over a Dedekind ring). For this, Wedhorn introduces the notion of scalar extension of category to define his Tannakian lattice. Wedhorn's Tannakian lattice is not necessarily equivalent to the full subcategory of rigid objects in a Tannakian category. In the next section we shall provide a characterization of such categories. In this subsection we will recall the notion of scalar extension of categories and some basis properties.

1.3.1. Let $\varphi : \mathbb{R} \longrightarrow S$ be a ring homomorphism. Let \mathbb{C} be an \mathbb{R} -linear category. The category \mathbb{C}_S obtained from \mathbb{C} by scalar extension φ is defined as follows. The objects of \mathbb{C}_S are the same as those of \mathbb{C} and for two objects X and Y in \mathbb{C}_S their hom-set is

$$\operatorname{Hom}_{\mathcal{C}_{S}}(X,Y) := \operatorname{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathbb{R}} S.$$

We have an S-linear category together with an R-linear functor $\varphi_* : \mathcal{C} \longrightarrow \mathcal{C}_S$.

If the map $\phi : R \longrightarrow S$ is flat, the functor ϕ_* preserves monomorphisms and epimorphisms [29, 3.6]. Let \mathcal{D} be another R-linear category and $\omega : \mathcal{C} \longrightarrow \mathcal{D}$ be an R-linear functor. Then we have a functor

 $\omega_{S}: \mathfrak{C}_{S} \longrightarrow \mathfrak{D}_{S}.$

If ϕ is flat and ω is faithful then so is ω_S [29, 3.7].

Assume that \mathcal{M} is an R-linear tensor category. The R-bilinear functor $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ extends to an S-bilinear functor $\otimes : \mathcal{M}_S \times \mathcal{M}_S \longrightarrow \mathcal{M}_S$. In this way \mathcal{M}_S is an S-linear tensor category. It is rigid if \mathcal{M} is rigid.

1.3.2. Let L be a flat R-coalgebra. Then $L_K := L \otimes_R K$ is a coalgebra over K. There is a natural functor from $Comod^o(L)$ to $Comod_f(L_K)$, $M \mapsto M \otimes_R K$. This induces a functor

$$\phi$$
 : Comod^o(L)_K \longrightarrow Comod^o(L_K).

This functor ϕ is an equivalence of abelian categories [29, Subsection 6.4]. Consequently, if G is a flat affine group scheme over R, we have an equivalence of abelian tensor categories

$$\operatorname{Rep}^{o}(G)_{K} \equiv \operatorname{Rep}_{f}(G_{K}).$$

1.3.3. Warning. Monomorphisms in Comod^o(L) are injective comodules maps, as the kernel of a map in Comod^o(L) is again in Comod^o(L) (L being flat). However an epimorphism in this category is not necessarily a surjective map, for instance a map $[a] : V \longrightarrow V$ given by multiplying with a non-unit a in R ($a \neq 0$) is an epimorphism in Comod^o(L) but is not surjective. Put in another way, the forgetful functor from Comod^o(L) to Mod(R) does not preserve epimorphisms. Later we shall impose the condition that our fiber functor preserves images (of morphisms) as a replacement for the exactness.

1.3.4. For an R-linear abelian tensor category C, we define the special fiber C_s at a closed point s of S := Spec(R) to be the full subcategory of objects which satisfy $m_s X = 0$, where m_s is the maximum ideal of R, which determines s.

For a flat affine group scheme G, we can identify $\operatorname{Rep}_{f}(G_{s})$ with $\operatorname{Rep}_{f}(G)_{s}$, cf. [17, Chapt. 10]. Indeed, let L = R[G] and $L_{s} := L \otimes_{R} R/m_{s}$. Then any L_{s} comodule is an L-comodule in a natural ways, as we have $V \otimes_{R} L \cong V \otimes_{k_{s}} L_{s}$ for any k_{s} -vector space V (on which R acts through the map $R \longrightarrow k_{s}$.

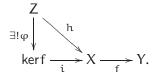
If C is Tannakian then the fiber functor ω yields an equivalence between C_s and $\text{Rep}_f(G_s)$ where G is the Tannakian group of C.

One can show that \mathcal{C}_s is equivalent to the scalar extension \mathcal{C}_{k_s} where $k_s = R/m_s$ is the residue field of R at s. In fact, the R-linearity yields the functor $X \mapsto X \otimes_R k_s$ from \mathcal{C} to \mathcal{C}_s and hence a functor from $\mathcal{C}_{k_s} \longrightarrow \mathcal{C}_s$ which is an equivalence, as the base change $R \longrightarrow k_s$ does not affect \mathcal{C}_s .

2. DUALITY FOR TANNAKIAN LATTICES

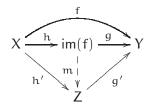
2.1. **The kernel and image of a morphism.** For the definition of a neutral Tannakian lattice, we shall need the notion of the kernel and the image of a morphism. The notion of kernels is standard in the category theory, we recall it here for the sake of the reader.

2.1.1. *Kernel.* Let C be an additive category, i.e. the hom-sets are equipped with abelian group structures and the composition of morphisms is bi-additive. For a morphism $f : X \longrightarrow Y$, the kernel of f is the equalizer of f and the zero map, i.e. the final object in the category of morphisms $h : Z \longrightarrow X$ satisfying $f \circ h = 0$:



The morphism $i : \ker f \longrightarrow X$ is then a monomorphism.

2.1.2. *Image-factorization*. The image-factorization of a morphism $f : X \longrightarrow Y$ is the initial object in the category of factorizations of the form $f = g \circ h$ with g being a monomorphism. That is, if f = g'h' is another factorization with g' being a monomorphis



then there exists a unique morphism m such that g = g'm. Since g' is a monomorphism, we will also have $m \circ h = h'$. If the image-factorization exists for any morphism we say that C is a *category with images*.

Let \mathcal{C}, \mathcal{D} be categories with images. A functor $\omega : \mathcal{C} \longrightarrow \mathcal{D}$ is said to preserve images if ω preserves the image-factorization for any morphisms in \mathcal{C} .

Example 2.1.3. (i) Any abelian category obviously has kernels and images. An exact functor between abelian categories preserves kernels and images.

(ii) The category $Mod^{o}(R)$ (R being a Dedekind ring) has kernels and images, which are determined set theoretically; the forgetful functor to $Mod_{f}(R)$ preserves kernels and images.

(iii) For a flat affine group scheme G over a Dedekind ring R, the category $\operatorname{Rep}^{o}(G)$ has kernels and images, and the forgetful functor from $\operatorname{Rep}^{o}(G)$ to $\operatorname{Mod}_{f}(R)$ preserves kernels and images.

2.2. Tannakian lattice.

2.2.1. In an additive tensor category the endomorphism ring of the unit object I is a commutative ring. Given a commutative ring R, an *additive rigid tensor category* \mathcal{T} *over* R, is an R-linear additive rigid tensor category in which the natural map $R \longrightarrow End(I)$ in an isomorphism, where I denotes the unit object.

An object J in \mathcal{T} is called trivial if there is a monomorphism $J \longrightarrow I^n$. The full subcategory of trivial objects of \mathcal{T} is denoted by \mathcal{T}^{triv} .

Definition 2.2.2. Let R be a Dedekind ring. A neutral Tannakian lattice over R is an additive rigid tensor category \mathcal{T} over R, in which kernels and images exist, the category \mathcal{T}_{K} given by scalar extension is abelian, and is equipped with an R-linear additive tensor functor $\omega : \mathcal{T} \longrightarrow Mod(R)$, satisfying the following conditions:

- T1) ω is faithful and preserves kernels and images.
- T2) ω restricted to T^{triv} is fully faithful.

2.2.3. Since \mathcal{T} is rigid, the image of ω is in $\mathsf{Mod}^{o}(\mathsf{R})$. Since ω is faithful, the hom-sets in \mathcal{T} are finite flat modules over R , consequently the functor $\iota_* : \mathcal{T} \longrightarrow \mathcal{T}_{\mathsf{K}}$ is faithful. Therefore, we shall identify \mathcal{T} with its image in \mathcal{T}_{K} . In particular, morphisms in \mathcal{T} are considered as morphisms in \mathcal{T}_{K} . An object X of \mathcal{T} , when considered as object of \mathcal{T}_{K} , will be denoted by X_{K} , and the image of a morphism f under ι_* will be denoted by the same symbol f.

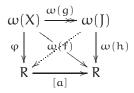
For an element $a \in R$ and objects X, Y in T, we shall use the symbol [a] to denote the morphism $X \longrightarrow Y$ induced by a. Further we shall write $[a] \circ f = f \circ [a]$ simply as af. If $f : X_K \longrightarrow Y_K$ is a morphism in \mathcal{T}_K , then there exists $a \in R$ (not uniquely determined) such that af is a morphism in T. The proof of the following lemma is obvious.

Lemma 2.2.4. Let $g : X \longrightarrow Y$ be such that $g : X_K \longrightarrow Y_K$ is an isomorphism. Then there exist $a \in R$ and $h : Y \longrightarrow X$, such that $h \circ g = g \circ h = [a]$. More general, given objects X, Y, Z in T and morphisms $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$. Assume that there exists $h : X_K \longrightarrow Y_K$ such that $g \circ h = f$. Then there exists $a \in R$ and $h' : X \longrightarrow Y$, such that ah = h' and $g \circ h' = af$.



Lemma 2.2.5. Let $f : X \longrightarrow Y$ be a morphism in \mathfrak{T} . If there exists a map $\varphi : \omega(X) \longrightarrow \omega(Y)$ such that $a\varphi = \omega(f)$ for some $a \in \mathbb{R}$, then there exists $g : X \longrightarrow Y$ with ag = f and $\omega(g) = \varphi$. In other words, the submodule $\omega(\operatorname{Hom}_{\mathfrak{T}}(X, Y))$ is saturated in $\operatorname{Hom}_{\mathbb{R}}(\omega(X), \omega(Y))$.

Proof. First, assume that Y = I – the unit object. Consider the image-factorization of f = hg: $X \longrightarrow J \longrightarrow I$. Applying ω we have the following commutative diagram



Thus $\omega(h)$ is injective and its image lies in aR, hence it factors as $\omega(h) : \omega(J) \xrightarrow{\psi} R \xrightarrow{\lfloor \alpha \rfloor} R$. Since ω is fully faithful on trivial objects, we have $\psi = \omega(h')$, that is, $\varphi = \omega(h' \circ g)$. Hence $f = a(h' \circ g)$.

The general case follows from this case by means of the isomorphism

$$\operatorname{Hom}_{\mathfrak{T}}(X,Y) \cong \operatorname{Hom}_{\mathfrak{T}}(X \otimes Y^{\vee},I)$$

which is compatible with the fiber functor ω .

Lemma 2.2.6. The functor $\omega_{K} : \mathfrak{T}_{K} \longrightarrow \text{Vect}(K)$ is exact. Thus $(\mathfrak{T}_{K}, \omega_{K})$ is a neutral Tannakian category over K.

Proof. Recall that the functor $i_* : T \longrightarrow T_K$ preserves mono- and epimorphisms.

Let $Y_K \xrightarrow{f} Z_K$ be an epimorphism in \mathcal{T}_K . Multiplying f with an element from R if needed, we can assume that f is in \mathcal{T} . Consider the image-factorization of f in \mathcal{T} : $f = Y \xrightarrow{h} im(f) \xrightarrow{g} Z$. Thus, in \mathcal{T}_K , g is both epi- and monomorphism, hence an isomorphism. Now $\omega(h)$ is surjective hence so is $\omega(h)_K$, thus $\omega(f)_K$ is also surjective.

Let $g: X_K \longrightarrow Y_K$ be the kernel of f in \mathcal{T}_K . We can similarly assume that g is in \mathcal{T} . Consider the image-factorization of the map $X \longrightarrow \ker(f)$ induced by g in $\mathcal{T}: X \xrightarrow{p} \operatorname{im}(g) \xrightarrow{q} \ker(f)$. We

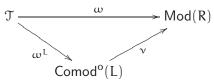
$$\omega(X) \xrightarrow{\omega(p)} \omega(\operatorname{im}(g)) \xrightarrow{\omega(q)} \omega(\operatorname{ker}(f)),$$

with $\omega(q)$ being invertible in Vect(K) and $\omega(p)$ being surjective. Consequently, $\omega(X)$ is the kernel of $\omega(f)$.

2.2.7. Let be G be a flat affine group scheme over R, then the category $\operatorname{Rep}^{o}(G)$ of representations of G in finite projective R-modules equipped with the forgetful functor to Mod(R), is a Tannakian lattice over R.

2.3. Tannakian duality.

2.3.1. Let (\mathfrak{T}, ω) be a Tannakian lattice over R. The discussion in 1.2.3 yields a coalgebra L and a factorization



where v is the forgetful functor. Recall that L is a bialgebra, ω^{L} is a tensor functor and we have an isomorphism

$$\operatorname{End}_{S}^{\otimes}(\omega \otimes S) \simeq \operatorname{Hom}_{R-\operatorname{Alg}}(L,S),$$

for any R-algebra S. According to [5, Prop. 1.13], we have

$$\operatorname{Aut}_{S}^{\otimes}(\omega \otimes_{R} S) \cong \operatorname{End}_{S}^{\otimes}(\omega \otimes_{R} S).$$

 $\operatorname{Aut}_{\widetilde{S}}(\omega \otimes_{\mathbb{R}} S) = \operatorname{End}_{\widetilde{S}}(\omega \otimes_{\mathbb{R}} S).$ Hence the functor $\operatorname{Aut}_{\mathbb{R}}^{\otimes}(\omega) : S \mapsto \operatorname{Aut}_{S}^{\otimes}(\omega \otimes_{\mathbb{R}} S)$ is representable by L. That is, L is a (commutative) Hopf algebra and tative) Hopt algebra and $\textbf{Aut}^\otimes_R(\omega)(S)\cong Hom_{alg}(L,S).$ Thus $\textbf{Aut}^\otimes_R(\omega)$ is an affine group scheme over R. tative) Hopf algebra and

$$\operatorname{Aut}_{\mathsf{R}}^{\otimes}(\omega)(\mathsf{S}) \cong \operatorname{Hom}_{\operatorname{alg}}(\mathsf{L},\mathsf{S})$$

Theorem 2.3.2. Let (\mathcal{T}, ω) be a Tannakian lattice over a Dedekind ring R. Then the group scheme $G = Aut_{R}^{\otimes}(\omega)$ is faithfully flat over R and ω induces an equivalence between T and Rep^o(G).

2.3.3. The difficulty lies in showing that L is flat. We shall use an indirect construction to prove that the Hopf algebra L satisfies the claims of Theorem 2.3.2. Lemma 2.2.6 shows that $\omega_{\rm K}: \mathfrak{T}_{\rm K} \longrightarrow {\rm Vect}({\rm K})$ is an exact, faithful functor, i.e. a fiber functor for the abelian rigid tensor category \mathcal{T}_{K} . Thus $(\mathcal{T}_{K}, \omega_{K})$ is a Tannakian category. The classical Tannakian duality yields a Hopf algebra \mathcal{L} over K and an equivalence

$$\omega_{\mathrm{K}}: \mathfrak{T}_{\mathrm{K}} \cong \mathrm{Comod}_{\mathrm{f}}(\mathcal{L}).$$

For each $X \in \mathcal{T}$, we have

$$\omega(X) \otimes_{\mathsf{R}} \mathsf{K} = \omega_{\mathsf{K}}(X_{\mathsf{K}}).$$

Thus

$$\omega(X)\otimes_{\mathsf{R}}\mathcal{L}=\omega_{\mathsf{K}}(X_{\mathsf{K}})\otimes_{\mathsf{K}}\mathcal{L}.$$

Therefore we can consider $\omega(X)$ as a comodule over the R-coalgebra \mathcal{L} . Denote by \mathcal{L}_R the union in \mathcal{L} of all the coefficient spaces $Cf(\omega(X))$ as X runs in the objects of \mathfrak{T} . Then \mathcal{L}_R is a Hopf R-subalgebra of \mathcal{L} , which coacts on all $\omega(X)$, (moreover we have $\mathcal{L}_R \otimes K = \mathcal{L}$). Thus the functor ω factors through the functor (followed by the forgetful functor):

(2)
$$\omega: \mathcal{T} \longrightarrow \text{Comod}^{o}(\mathcal{L}_{R}).$$

2.3.4. *Proof of Theorem 2.3.2.* We proceed by showing that the functor ω in (2) is an equivalence of categories.

 ω is fully faithful: By assumption, ω is faithful. On the other hand, the Tannakian duality for T_K says that

$$\omega(\operatorname{Hom}_{\mathfrak{T}}(X,Y)) \otimes_{\mathbb{R}} \mathsf{K} = \operatorname{Hom}^{\mathcal{L}}(\omega_{\mathsf{K}}(X_{\mathsf{K}}),\omega_{\mathsf{K}}(Y_{\mathsf{K}})).$$

Let $\varphi \in \text{Hom}^{\mathcal{L}_R}(\omega(X), \omega(Y))$. Then the above equality ensures the existence of a morphism $f \in \text{Hom}_{\mathcal{T}}(X, Y)$ such that $\omega(f) = a\varphi$ for some $a \in R$. Now Lemma 2.2.5 implies that there exists $g \in \text{Hom}_{\mathcal{T}}(X, Y)$ such that $\omega(g) = \varphi$. That is, $\omega : \mathcal{T} \longrightarrow \text{Comod}^o(\mathcal{L}_R)$ is full.

 ω is essentially surjective: Each finite projective \mathcal{L}_R -comodule M is a special subquotient of \mathcal{L}_R^r (direct sum of r copies of \mathcal{L}_R) by means of the coaction: $\delta : M \longrightarrow M \otimes_R \mathcal{L}_R$, see 1.1.5. Consequently for any subcomodule N in \mathcal{L}_R^r containing M, the quotient N/M is also flat. In particular, we can choose N to be the subcomodule $Cf(\omega(X))^r$ for some X in T. Thus we conclude that M is a special subquotient of an object in the image of ω .

Next, we show that each N in Comod^o(\mathcal{L}_R) is isomorphic in this category to some $\omega(X)$. First assume that N is a quotient of some $\omega(Y)$:

$$0 \longrightarrow M \longrightarrow \omega(Y) \longrightarrow N \longrightarrow 0.$$

According to 1.3.2, there exists some Z in \mathcal{T} such that $N \otimes K \cong \omega_K(Z_K)$ in $Comod(\mathcal{L}_K)$. Such an isomorphism yields an injective map $N \longrightarrow \omega(Z)$ in Comod(L). Consider the composition $\omega(Y) \longrightarrow N \longrightarrow \omega(Z)$. Since ω is fully faithful, this map is the image of a morphism $f: Y \longrightarrow Z$. By assumption, f has image in \mathcal{T} and ω preserves it, hence we claim $\omega(im(f)) = N$. Dualizing the above sequence we have an exact sequence

$$0 \longrightarrow N^{\vee} \longrightarrow \omega(Y^{\vee}) \longrightarrow M^{\vee} \longrightarrow 0.$$

The same discussion shows that $M^{\vee} \cong \omega(\operatorname{img})$ for some morphism g in \mathfrak{T} , thus $M \cong \omega(T)$ with $T := (\operatorname{imf})^{\vee}$. Thus, all special subobjects of $\omega(X)$ are also in $\operatorname{im}(\omega)$. Consequently all special subquotients of ω are in $\operatorname{im}(\omega)$. This completes the proof of the fact that ω is an equivalence of categories between \mathfrak{T} and $\operatorname{Comod}^{o}(\mathcal{L}_{R})$.

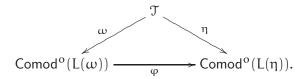
 \mathcal{L}_R is the coend of ω : this follows from the equivalence just established. In deed, by the equivalence above, $\omega : \mathcal{T} \longrightarrow Mod(R)$ can be identified with the forgetful functor $Comod^o(\mathcal{L}_R) \longrightarrow Mod(R)$. But the coend of this last functor is just \mathcal{L}_R , cf. 2.3.1. This finishes the proof of Theorem 2.3.2.

2.4. **Torsors.** Let (\mathfrak{T}, ω) be a Tannakian lattice. Let S be a faithfully flat R-algebra. A fiber functor $\eta : \mathfrak{T} \longrightarrow \mathsf{Mod}^o(S)$ is defined to be an R-linear tensor functor satisfying the following conditions:

(T1) η is faithful and preserves kernels and images.

In this section we construct out of this data a torsor $\mathbf{Iso}_{R}^{\otimes}(\omega,\eta)$ over $\mathbf{Aut}_{R}^{\otimes}(\omega)$. In particular we will show that for different neutral fiber functors ω and ω' , the representation categories of $\mathbf{Aut}_{R}^{\otimes}(\omega)$ and $\mathbf{Aut}_{R}^{\otimes}(\omega')$ are equivalent, thus determining a unique *abelian envelope* of \mathcal{T} .

2.4.1. *The abelian envelope of* \mathfrak{T} . Let (\mathfrak{T}, ω) be a neutral Tannakian lattice and let $\eta : \mathfrak{T} \longrightarrow Mod(R)$ be another fiber functor. Let $L(\omega)$ be the coend of ω and $L(\eta)$ be the coend of η . Then Tannakian duality for (\mathfrak{T}, ω) and (\mathfrak{T}, η) yields a functor $\varphi : Comod^{\circ}(L(\omega)) \longrightarrow Comod^{\circ}(L(\eta))$:



The subcategory of trivial objects in $Comod^o(L(\omega))$ is exactly the category $Mod^o(R)$ of finite projective R-modules. The restriction of φ to $Mod^o(R)$ is thus a faithful functor, preserving kernels and images and sending R to itself. It is therefore a fully faithful functor. Thus, applying Theorem 2.3.2 to φ we conclude that this restriction of φ is an equivalence of categories. We will show that φ extends to an equivalence between $Comod(L(\omega))$ and $Comod(L(\eta))$.

2.4.2. Let V be in Comod^o(L(ω)). Denote C := Cf(V) \subset L(ω). Then Comod^o(C) is equivalent to $\langle V \rangle_s$, cf. 1.1.7. Let $W := \eta(V) \in \text{Comod}^o(L(\eta))$ and D := Cf(W) \subset L(η). Then φ induces an equivalence between Comod^o(C) and Comod^o(D).

Let A := Hom(C, R), then A is a (non-commutative) algebra, finite projective as an R-module and we have an equivalence $Comod(C) \cong Mod(A)$ inducing and equivalence

$$Comod^{o}(C) \cong Mod^{o}(A).$$

Here, we denote by Mod(A) the category of finite left A-modules and by $Mod^{o}(A)$ full subcategory modules which are finite projective over R. Thus, denoting B := Hom(D, R), we have and equivalence, also denoted by φ :

$$\mathsf{Mod}^{\mathbf{o}}(\mathsf{A}) \xrightarrow{\Phi} \mathsf{Mod}^{\mathbf{o}}(\mathsf{B}).$$

Lemma 2.4.3. The functor φ extends to an equivalence between $Mod_f(A)$ and $Mod_f(B)$.

Proof. Since φ preserves kernels and images, it preserves exact sequences in Mod^o(A) (i.e sequences in Mod^o(A), exact in Mod(A).

Denote $P := \varphi(A)$. Then P is a B-A-bimodule (the action of A on P is induced from the right action of A on itself, which commutes with morphisms in Mod(A)). For an object $M \in Mod^{o}(A)$, consider a finite resolution $A^{m} \longrightarrow A^{n} \longrightarrow M \longrightarrow 0$. We have an exact sequence

$$\varphi(A^m) \longrightarrow \varphi(A^n) \longrightarrow \varphi(M) \longrightarrow 0$$

in Mod(B). Since φ is additive, we have $\varphi(A^n) = P^n$, hence we have canonical isomorphism

$$\varphi(\mathsf{M}) \cong \mathsf{P} \otimes_{\mathsf{A}} \mathsf{M}.$$

Thus ϕ coincides with $P \otimes_A -$ on $Mod^o(A)$.

We show that P is flat over A. For a finite A-module M, consider a resolution

 $0 \longrightarrow N \longrightarrow A^m \longrightarrow M \longrightarrow 0.$

Since N is R-projective being a submodule of A^m , and since $\varphi(-) = P \otimes - : Mod^o(A) \longrightarrow Mod(B)$ preserves monomorphisms, the long exact sequence

$$0 \longrightarrow \operatorname{Tor}_{1}^{A}(\mathsf{P},\mathsf{M}) \longrightarrow \mathsf{P} \otimes \mathsf{N} \longrightarrow \mathsf{P}^{\mathfrak{m}} \longrightarrow \mathsf{P} \otimes_{A} \mathsf{M} \longrightarrow 0$$

shows that $\operatorname{Tor}_1^A(P, M) = 0$. Thus we have an exact functor $P \otimes -: \operatorname{Mod}_f(A) \longrightarrow \operatorname{Mod}_f(B)$, which reduces to an equivalence $\operatorname{Mod}^o(A) \longrightarrow \operatorname{Mod}^o(B)$. Hence it is itself an equivalence from $\operatorname{Mod}_f(A)$ to $\operatorname{Mod}_f(B)$.

Proposition 2.4.4. *The functor* φ : Comod^o(L(ω)) \longrightarrow Comod^o(L(η)) *extends to an equivalence between* Comod_f(L(ω)) *and* Comod_f(L(η)).

Proof. The equivalence is obtained by extending $\langle V \rangle_s$ larger and larger.

2.4.5. The torsor $\mathbf{Iso}_{\mathsf{R}}^{\otimes}(\omega,\eta)$. Consider now the more general situation: η is a fiber functor $\mathcal{T} \longrightarrow \mathsf{Mod}(S)$, where S is an R-algebra. Recall that $\mathbf{Iso}_{\mathsf{R}}^{\otimes}(\omega,\eta)$ is the functor which associates to each S-algebra S' the set $\mathbf{Iso}^{\otimes}(\omega,\eta)(S')$ of natural isomorphisms compatible with the tensor structure between the two given functors. According to [5, Prop. 1.13], this set is equal to the set $\mathrm{Nat}_{\mathsf{S}'}^{\otimes}(\mathsf{S}' \otimes_{\mathsf{R}} \omega,\mathsf{S}' \otimes_{\mathsf{S}} \eta)$. The algebra $\mathsf{L}(\omega,\eta)$ represents functor $\mathbf{Iso}_{\mathsf{R}}^{\otimes}(\omega,\eta)$ if it satisfies

$$\operatorname{Nat}_{S'}^{\otimes}(\omega \otimes_{\mathbb{R}} S', \eta \otimes_{S} S') \cong \operatorname{Hom}_{S-\operatorname{alg}}(L(\omega, \eta), S'),$$

for any S-algebra S'. As in 1.2.3, we notice that $L(\omega, \eta)$ can be defined as the S-module representing the functor

$$S' \mapsto \operatorname{Nat}_{S'}(\omega \otimes_R S', \eta \otimes_S S') \cong \operatorname{Nat}_R(\omega, \eta \otimes_S S').$$

That is, we determine $L(\omega, \eta)$ by the functorial isomorphism

(3)
$$\operatorname{Nat}_{R}(\omega, \eta \otimes_{S} N) \cong \operatorname{Hom}_{S}(L(\omega, \eta), N),$$

for any S-module N.

2.4.6. Identifying \mathcal{T} with Comod^o(L) by means of ω , we can consider ω as the identity functor and η as a tensor functor Comod^o(L) $\longrightarrow Mod(S)$, where $L := L(\omega)$.

Let $C \subset L$ be an R-finite subcoalgebra, and set $A = \text{Hom}_R(C, R)$. The discussion in the proof of Lemma 2.4.3 shows that the restriction of η to $\text{Comod}^o(C) = \text{Mod}^o(A)$ extends to an exact functor $P \otimes_A - : \text{Mod}_f(A) \longrightarrow \text{Mod}_f(S)$, where $P = \eta(A)$.

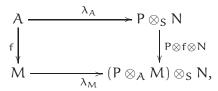
Lemma 2.4.7. Consider ω and η as functors on the category Mod^o(A), we have natural isomorphism

$$\operatorname{Nat}_{\mathbb{R}}(\omega, \eta \otimes_{S} \mathbb{N}) \cong \mathbb{P} \otimes_{S} \mathbb{N} \cong \operatorname{Hom}_{S}(\eta(\mathbb{C}), \mathbb{N})$$

for any S-module N.

Proof. This is standard. By the functoriality, a natural transformation $\lambda : \omega \longrightarrow \eta \otimes_S N$ is uniquely determined by its value at A, i.e. an R-linear map $\lambda_A : A \longrightarrow \eta(A) \otimes_S N = P \otimes_S N$,

which is A-linear. Indeed, this is a consequence of the following commutative diagram, for any f in $Mod^{o}(A)$:



forcing $\lambda_M = \lambda_A \otimes_A M$. (The A-linearity follows from the choice M = A). Conversely, any such A-linear map determines a natural equivalence.

As to the last isomorphism, notice that C is R-finite projective, hence A is the dual of C in $Comod^{o}(L)$, therefore $P = \eta(A)$ is dual to $\eta(C)$ as S-modules. Thus we have

$$P \otimes_{S} N = Hom_{S}(\eta(C), S) \otimes_{S} N \cong Hom_{S}(\eta(C), N)$$

This finishes the proof.

Proposition 2.4.8. The scheme $Iso_{R}^{\otimes}(\omega, \eta)$ is a torsor under the group scheme $Aut_{R}^{\otimes}(\omega)$.

Proof. This is a remedy of [5, Prop. 3.2]. According to the isomorphism in (3) and the lemma above, the S-module

$$L(\omega,\eta) = \lim_{C \subset L(\omega)} \eta(C)$$

represents the functor $\mathbf{Iso}_{\mathsf{R}}^{\otimes}(\omega,\eta)$. Note that the transition maps in the directed system are inclusions of subcoalgebras $C \hookrightarrow C'$ of L, which give rise to injective maps $\eta(C) \longrightarrow \eta(C')$. The torsor action is obvious, loc.cit. It remains to check that $\mathbf{Iso}_{\mathsf{R}}^{\otimes}(\omega,\eta)$ is faithfully flat over S, i.e. to check that $L(\omega,\eta)$ is faithfully flat over S. As $L(\omega,\eta)$ is the direct limit of S-projective modules, it is flat over S. We show the faithfulness.

If a finite subcoalgebra C contains the unit element of $L(\omega)$, then the inclusion $R \longrightarrow C$ splits in Mod(R) (by means of the counit). We obtain an exact sequence in Comod^o(C):

$$0 \longrightarrow R \longrightarrow C \longrightarrow C/R \longrightarrow 0$$

giving rise to an exact sequence in $Comod^{o}(S)$ (as ω preserves kernels and images)

$$0 \longrightarrow S \longrightarrow \eta(C) \longrightarrow \eta(C/R) \longrightarrow 0.$$

Passing to the limit we obtain an inclusion $S \longrightarrow L(\omega, \eta)$, which is pure, as

$$L(\omega,\eta)/R \cong \varprojlim_{R \subset C \subset L} \eta(C/R)$$

is flat. This shows that $L(\omega, \eta)$ is faithfully flat over S.

3. COALGEBRA HOMOMORPHISMS

3.1. **Specially locally finite coalgebras.** An important property of coalgebras over a field is the local finiteness: a coalgebra is the union of its finite dimensional subcoalgebras. This property formally generalizes to flat coalgebras over a Dedekind ring. Indeed, let $M \subset L$ be a finite subcomodule, then M is projective over R and is contained in the subcoalgebra Cf(M) of L (see proof of Lemma 3.1.2).

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However this is not fully reflected in the Tannakian duality. For a subcoalgebra C of coalgebra L over a field k, the category $Comod_f(C)$ can be identified with a full, exact subcategory of $Comod_f(L)$, which is closed under taking subobjects. This is no more the case for flat coalgebras over a Dedekind ring as the example 3.1.3 below shows. The reason is that the quotient module L/C may be non-flat over the base ring. If we try to enlarge C so that the quotient becomes flat, we cannot guarantee that it remains finite over R. In view of Tannakian duality, this reflects the fact that the full abelian subcategory generated by a single object in a comodule category may become quite large.

This phenomenon is one of the main obstructions to the study of flat coalgebras and flat group schemes over a Dedekind ring. Below we will show that the coordinate ring of a reduced, connected group scheme is specially locally finite as a coalgebra (Proposition 3.1.7).

Recall that a subcomodule M of an L-comodule N is said to be special if N/M is flat over R; a special subquotient M of an L-comodule N is a special submodule of a quotient of N, or, equivalently, a quotient of a special submodule of N.

Definition 3.1.1. Let L be an R-flat coalgebra.

- (i) A comodule N is said to be *specially locally finite* if any finite subcomodule of N is contained in an R-finite special subcomodule;
- (ii) A subcoagebra C of L is said to be *special* if L/C is flat. A homomorphism of flat coalgebras f : L' \longrightarrow L is said to be special if f(L') is a special subcoalgebra of L;
- (iii) L is said to be *specially locally finite* if for any finite subcomodle C, there exists a finite special subcoalgebra containing C^1 .

Let N be an L-comodule. Then N_{tor}, the R-torsion submodule of N is an L-subcomodule. Hence for any L-subcomodule M, the preimage of $(N/M)_{tor}$ in N, denoted M^{sat} , is an L-comodule. Since R is a Dedekind ring, the quotient N/M^{sat} is flat, being torsion free. Thus M^{sat} is the smallest special subcomodule of N, containing M. It is called the saturation of M in N.

Lemma 3.1.2. An R-flat coalgebra L is specially locally finite if and only if it L is locally finite as a comodule on itself.

Proof. Assume that L is specially locally finite as a right comodule on itself. Let C be a finite subcoalgebra of L and C^{sat} the saturation. It is to show that C^{sat} is a subcoalgebra.

We have the filtration $C^{sat} \otimes C^{sat} \subset C^{sat} \otimes L \subset L \otimes L$, the successive quotients of which are flat, hence $L \otimes L/C^{sat} \otimes C^{sat}$ is also flat. Thus

$$(C \otimes C)^{sat} \subset C^{sat} \otimes C^{sat}$$
.

Hence, by the definition of C^{sat}, we have

$$\Delta(C^{sat}) \subset (C \otimes C)^{sat} \subset C^{sat} \otimes C^{sat}.$$

For the converse statement we use the well-known fact that each finite subcomodule of L is contained in a finite subcoalgebra of L. Indeed, given $M \subset L$ finite, then it is R-projective and

¹Specially locally finite coalgebras are called IFP coalgebras in [14, I.3.11].

the coaction $M \longrightarrow M \otimes L$ induces a coalgebra map (cf. 1.1.4)

$$M^{\vee}\otimes M \longrightarrow L, \quad \phi \otimes \mathfrak{m} \mapsto \sum \phi(\mathfrak{m}_i)\mathfrak{m}'_i, \quad \text{where } \Delta(\mathfrak{m}) = \sum_i \mathfrak{m}_i \otimes \mathfrak{m}'_i.$$

Its image is the coefficient space Cf(M) of M, which contains M. In fact, let ε_M be the restriction of ε to M, then $\varepsilon_M \otimes \mathfrak{m} \mapsto \sum_i \varepsilon(\mathfrak{m}_i)\mathfrak{m}'_i = \mathfrak{m}$.

The next example shows that there exist coalgebras which are not specially locally finite.

Example 3.1.3 ([7], Remarque 11.10.1). Let $0 \neq x \in R$ be a non-unit element. Let G be the affine group scheme over R determined by the Hopf subalgebra of $K[\mathbf{G}_{\alpha}] = K[T]$:

$$R[G] := \{ P \in K[T] | P(0) \in R \},\$$

that is, R[G] consists of polynomials in K[T] with the constant coefficients belonging to R. Let C_i be the subcoalgebra spanned (over R) by 1 and $x^{-i}T$. Then we have

$$C_i \subsetneq C_{i+1}, \quad xC_{i+1} \subsetneq C_i.$$

Thus the saturation of C_0 is not finite.

In what follows we will need some standard facts on tensor products and flat modules, cf. [4].

Lemma 3.1.4. Let $A \subset B$ be flat R-modules and $M, M_1, M_2 \subset N$ be arbitrary R-modules. Then

- (i) $M_1 \otimes A \cap M_2 \otimes A = (M_1 \cap M_2) \otimes A$, $M_1 \otimes A + M_2 \otimes A = (M_1 + M_2) \otimes A$, as subsets of $N \otimes A$;
- (ii) if B/A is also flat, we have $N \otimes A \cap M \otimes B = M \otimes A$ as submodules in $N \otimes B$.

The following result is proved in more generality in [14, I.3.11], we recall it here for completeness. The reader is referred to [26, Tag 0599] for the notion of Mittag-Leffler system.

Proposition 3.1.5. Let R be a Dedekind ring and L be an R-flat coalgebra.

- (i) If L is specially locally finite then L is Mittag-Leffler as an R-module. Hence, if L is moreover countably generated over R, it is R-projetive.
- (ii) If L is R-projective, then it is specially locally finite as an R-coalgebra.

Proof. (i) Let $\{C_{\alpha}\}$ be the directed system of finite special subcoalgebras. Then for any finite R-module N, the system $\text{Hom}_{R}(C_{\alpha}, N)$ is Mittag-Leffler. In fact, each inclusion $C_{\alpha} \longrightarrow C_{\beta}$ splits, as C_{β}/C_{α} is R-torsion free and finite, hence projective over R. Consequently the map

$$\operatorname{Hom}_{R}(C_{\beta}, N) \longrightarrow \operatorname{Hom}_{R}(C_{\alpha}, N)$$

is surjective. Thus by definition, L is Mittag-Leffler as an R-module. It is well-known that a flat, countably generated, Mittag-Leffler module is projective.

(ii) We show that any projective comodule is specially locally finite. If $N \subset M$ is a subcomodule then N^{sat} is the preimage of $(M/N)_{\text{tor}}$ under the quotient map $M \longrightarrow M/N$. Thus we have to show that N^{sat} is finite provided that M is projective and N is finite as R-modules. This is a pure question of R-modules. Embed M is a free R-module F as a direct summand. Replacing M by F will only enlarge N^{sat} , thus we can assume that M is free over R. Then, as N is finite,

we can find a free direct summand of F_0 which contains N. Now $F_0 = F_0^{sat}$ implying $N^{sat} \subset F_0$, hence it is finite.

Questions 3.1.6. It is not known if any specially locally finite R-flat coalgebra is R-projective. Another interesting question is: which affine flat R-group scheme of finite type is specially locally finite.

Proposition 3.1.7. Let G be a flat group scheme of finite type over a Dedekind ring R. Assume that the generic fiber G_K is reduced and connected. Then R[G] is specially locally finite as an R-coalgebra.

Proof. Let I be the augmented ideal of R[G], that is $I = \ker(\varepsilon)$. Since the R[G] is flat over R, the map $R[G] \longrightarrow R[G] \otimes_R K = K[G_K]$ is injective and as K is flat over R, the augmentation ideal of $K[G_K]$ is $I_K = I \otimes_R K$. With the assumption that G_K is reduced and connected, $K[G_K]$ is an integral domain. By Krull's intersection theorem $\bigcap_m (I_K)^m = 0$.

Let $M \subset R[G]$ be a finite submodule. Then there exists m such that $M \otimes K \cap (I_K)^m = 0$. We have $(I_K)^m = I^m \otimes K$. Hence $(M \cap I^m) \otimes K = 0$ implying $M \cap I^m = 0$. It follows that $M^{sat} \cap I^m = 0$. Indeed, if $0 \neq a \in M^{sat} \cap I^m$ then there exists $0 \neq r \in R$ such that $ra \in M \cap I^m$, it forces ra = 0, consequently a = 0 as R[G] is torsion free. Thus, the map $M^{sat} \longrightarrow R[G]/I^m$ is injective. But the module $R[G]/I^m$ is finite, hence so is M^{sat} .

On the other hand, the situation for group schemes with finite fibers turns out to be more complicated, as in the following example, communicated to us by dos Santos.

Example 3.1.8. Let R be a DVR of equal positive characteristic p, with uniformizer π . Let G be the group scheme determined by the Hopf algebra

$$R[G] := R[T]/(\pi T^p - T), \quad \Delta(T) = 1 \otimes T + T \otimes 1.$$

Then the fibers of G are étale group scheme but R[G] is not specially locally finite. Indeed, the saturation of the finite subcomodule spanned by 1 and T contains T^{p^k} , $k \ge 1$, and is not finite.

3.2. Tannakian description of homomorphisms of coalgebras. In the second part of this paragraph we will use the Tannakian duality to characterize (special) injective and surjective homomorphisms of flat coalgebras.

Let $f: L' \longrightarrow L$ be homomorphism of flat coalgebras over a Dedekind ring R. We denote by $\omega_f: \text{Comod}_f(L') \longrightarrow \text{Comod}_f(L)$ the "restriction" functor (which considers each L'-comodule as an L-comodule by means of f, thus ω_f is the identity functor on the underlying R-modules). Then $\omega_f^{\circ}: \text{Comod}^{\circ}(L') \longrightarrow \text{Comod}^{\circ}(L)$ will denote the restriction of ω_f to the subcategory of finite projective comodules.

Proposition 3.2.1. Let $f : L' \longrightarrow L$ be homomorphism of flat coalgebras over a Dedekind ring R. Then

- (i) f is injective if and only if the functor ω_f° is fully faithful and its image in $Comod_f(L)$ is closed under taking special subobjects.
- (ii) f is injective and special if and only if the natural functor ω_f° is fully faithful and its image in $Comod_f(L)$ is closed under taking subobjects. In this case, the functor ω_f is also fully faithful and its image is closed under taking subobjects.

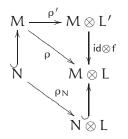
Proof. We shall show the "only if" implication for both claims at once. First notice that the full faithfulness claim in (i) is essentially proved in Lemma 1.1.7. For claim (ii) the proof is almost the same: Assume that f is injective and special. Then the functor $\omega_f : \text{Comod}_f(L') \longrightarrow \text{Comod}_f(L)$ can be considered as the identity functor on the underlying module category. Hence it is obviously faithful (and so is ω_f°).

As mentioned in the proof of Lemma 1.1.7, the condition for a map $\varphi : M \longrightarrow N$ to be L'-comodules reads as follows: $\rho'_N \varphi - (\varphi \otimes id) \rho'_M : M \longrightarrow N \otimes L'$ is the zero map:

$$\begin{array}{c} M \xrightarrow{\rho'_{M}} M \otimes L' \xrightarrow{\operatorname{id} \otimes f} M \otimes L \\ \varphi & \downarrow & \varphi \otimes \operatorname{id} \downarrow & \downarrow \varphi \otimes \operatorname{id} \\ N \xrightarrow{-\rho'_{N}} N \otimes L' \xrightarrow{-\operatorname{id} \otimes f} N \otimes L \end{array}$$

Now, if L/L' is flat over R, the horizontal map $id \otimes f : N \otimes L' \longrightarrow N \otimes L$ in the above diagram is injective. Hence the map $\rho'_N \varphi - (\varphi \otimes id) \rho'_M$ is zero if and only if its composition with $id \otimes f$, which is $\rho_N \varphi - (\varphi \otimes id) \rho_M : M \longrightarrow N \otimes L$, is zero. Thus ω_f is full.

We show the closedness under taking (special) subquotients. For $(M, \rho') \in \text{Comod}_f(L')$, it image under ω_f is denoted by (M, ρ) . Let (N, ρ_N) be a sub L-comodule of (M, ρ) . Thus we have commutative diagram



To show that ρ_N comes from a coaction of L' on Y amounts to showing that $\rho_N(N) \subset N \otimes f(L')$. The above diagram shows that $\rho_N(N) \subset N \otimes L \cap M \otimes f(L')$. If either M, N, M/N or L/f(L') are flat over R, according to Lemma 3.1.4, one has equality

(4)
$$N \otimes L \cap M \otimes f(L') = N \otimes f(L').$$

Thus $\rho_N(N) \subset N \otimes f(L')$, that is, ρ' restricts to a coaction of L' on N.

Conversely, assume that the functor $\omega_f^o : Comod(L')^o \longrightarrow Comod(L)^o$ is fully faithful and its image is closed under taking special subobjects. By the flatness of K over R, the functor $Comod_f(L')_K \longrightarrow Comod_f(L)_K$ is fully faithful and preserves mono- and epimorphisms. Therefore, by the equivalence in 1.3.2 the functor $Comod(L' \otimes K) \longrightarrow Comod(L \otimes K)$ satisfies conditions of [5, Thm 2.21], hence $L' \otimes K \longrightarrow L \otimes K$ is injective. Consequently the map $f : L' \longrightarrow L$ is injective, (i) is proved.

Now assume that the image of ω_f^{o} is closed under taking subobjects. The discussion above shows that f is injective. We will identify L' with a subcoalgebra of L. For a non-invertible $0 \neq \pi \in \mathbb{R}$, denote $C := \pi L \cap L' \supset \pi L'$. By means of Lemma 3.1.4 we see that C is also an L-subcomodule of L'. Since an L-comodule is the union of its finite subcomodule, the assumption

on ω_f^{o} implies that C is in fact an L'-subcomodule of L':

$$\Delta(\mathsf{C}) \subset \mathsf{C} \otimes \mathsf{L}' \subset \pi \mathsf{L} \otimes \mathsf{L}'.$$

Thus for any $c \in C$ we have

$$\Delta(c) = \sum \pi a_i \otimes b_i, \quad a_i \in L, \ b_i \in L'.$$

Hence $c = \sum_i \pi \epsilon(a_i) b_i \in \pi L'$. That is $C \subset \pi L'$, consequently $\pi L \cap L' = \pi L'$. The last equation holds for any $\pi \in R$, it follows that L/L' is torsion free over R, hence flat, as R is a Dedekind ring.

Remarks 3.2.2. The proof of Proposition 3.2.1 is based on the fact that, when R is a field, the map $Coend(\omega|\langle X \rangle) \longrightarrow L$ is injective. The reader is referred to [27, Thm 6.4.4] for the detailed proof of this fact or to [16, Lem 1.2] for a short proof. In Saavedra's proof of this fact [21, II.2.6.2.1], the proof of implication d) \Rightarrow a) is incomplete.

We now express the local finiteness in terms of Tannakian duality. First we have the following characterization of finite coalgebras.

Proposition 3.2.3. Let L be a flat coalgebra over a Dedekind ring R. Then L is finite over R if and only if $Comod_f(L)$ has a projective generator which is R-projective.

Proof. Assume that L is finite. Then L is projective over R and there is an equivalence between Comod(L) and $Mod(L^{\vee})$, where L^{\vee} is the dual R-module to L. Now L^{\vee} is a projective generator in $Comod_f(L)$.

Conversely, assume that $Comod_f(L)$ has a projective generator, say P, which is R-projective. Then this category is equivalent to $Mod_f(A)$, where $A = End^L(P)$. As $End_R(P)$ is R-finite projective and A is an R-submodule of $End_R(P)$, it is also projective. Therefore A^{\vee} is a finite flat R-coalgebra and Mod(A) is equivalent to $Comod(A^{\vee})$. By Tannakian duality we conclude that $A^{\vee} \cong L$, whence L is finite.

Recall that for a comodule X of L, $\langle X \rangle$ denotes the full subcategory of Comod_f(L) consisting of subquotients of finite direct sums of copies of X. If X is R-finite projective then $\langle X \rangle$ satisfies the conditions of Definition 1.2.1 (since a submodule of a finite projective R-module is again finite projective, we can take for $\langle X \rangle^{\circ}$ the full subcategory of subobjects of finite direct sums of copies of X). Hence, by means of Theorem 1.2.2, we obtain a coalgebra Coend($\omega | \langle X \rangle$), the comodule category of which is equivalent to $\langle X \rangle$.

Proposition 3.2.4. Let L be a flat coalgebra over a Dedekind ring R and let ω denote the forgetful functor Comod_f(L) \longrightarrow Mod(R).

- (i) For each R-finite projective comodule X of L the natural map $Coend(\omega|\langle X \rangle) \longrightarrow Coend(\omega) =$ L is injective and special. If L is specially locally finite then $Coend(\omega|\langle X \rangle)$ is finite over R.
- (ii) If for any R-finite projective comodule X, $Coend(\omega|\langle X \rangle)$ is finite, then L is specially locally finite.

Proof. (i) The first claim follows from Tannakian duality Theorem 1.2.2 and Proposition 3.2.1.

Assume now that L is specially locally finite. Let M be an R-finite projective comodule of L. Then by assumption on L, there exists a special subcoalgebra C of L, which is finite over

R and such that the coaction of L on M factors though that of C. According Proposition 3.2.1 (ii), $Comod_f(C)$ is a full subcategory of $Comod_f(L)$ closed under taking subobjects. Since $M \in Comod_f(C)$, we have $\langle M \rangle \subset Comod_f(C)$. Hence we have a factorization

$$Coend(\omega|\langle M \rangle) \longrightarrow C \longrightarrow L.$$

But C is R-finite, whence the claim.

(ii) According to (i), we can cover L by its finite special subcoalgebras $L=\bigcup L_i,$ where $\{L_i\}$ is

a cofinal directed system, i.e. any two coalgebras L_i, L_j are contained in some L_k . Hence, if C is a finite subcoalgebra of L, then C is contained in some L_i . Since L_i is saturated in L, $C^{sat} \subset L_i$, hence is finite.

Finally we provide a condition for the surjectivity of a colagebra homomorphism.

Proposition 3.2.5. Let $f : L' \longrightarrow L$ be a morphism of flat coalgebras over R. Then f is surjective if and only if the induced functor $\omega_f^{\circ} : \text{Comod}^{\circ}(L') \longrightarrow \text{Comod}^{\circ}(L)$ satisfies the following condition: each $M \in \text{Comod}^{\circ}(L)$ is a special subquotient of $\omega_f(N)$ for some $N \in \text{Comod}^{\circ}(L')$.

Proof. Assume that f is surjective. Let M be an R-finite projective L-comodule. Then M is a special subcomodule of $L^{\oplus r}$ for some r > 0 (cf. 1.1.5). We identify M with a subcomodule of $L^{\oplus r}$ and choose a generating set $\{m_i\}$ of M. Let $m'_i \in L'^{\oplus r}$ be such that $f(m'_i) = m_i$. The set $\{m'_i\}$ is contained in some finite module N of $L'^{\oplus r}$. The homomorphic image of N in $L^{\oplus r}$ is denoted by N₁, this is an L-subcomodule of $L^{\oplus r}$ which contains M. Since M is special in $L^{\oplus r}$, it is also special in N₁. Thus M is a special subcomodule of the quotient N₁ of $\omega_f(N)$, where N is an L'-comodule.

Conversely, assume that ω_f^o has the stated property. It suffices to show that any R-finite subcoalgebra C of L is in the image of f. Consider such a C as a (right) L-comodule. By assumption, there exists an R-finite projective L'-comodule N, such that C is a special subquotient of $\omega_f(N)$ in Comod_f(L). Since C = Cf(C), and C is a special subquotient of N, according to subsection 1.1.5 we conclude that $C \subset Cf(N)$. On the other hand, it follows from the construction that Cf(N) is the image of $Cf_{L'}(N)$ - the coefficient space of N considered as an L'-comodule. Thus the map L' \longrightarrow L is surjective.

4. TANNAKIAN DESCRIPTION OF GROUP SCHEME HOMOMORPHISMS

Let R be a Dedekind ring with fraction field K. Let $f : G \longrightarrow G'$ be a homomorphism of flat affine group schemes over R. We say that f is surjective or a quotient homomorphism if it is faithfully flat. In the first part of this chapter we give a necessary and sufficient condition for the faithful flatness of f, in terms of Tannakian duality. Then we will give a condition for f to be a closed immersion. Finally we give a criterion for the exactness of a sequence of group homomorphisms.

4.1. **Faithfully flat homomorphisms and closed embedding.** The following theorem is a generalization of the well-known faithful flatness theorem for Hopf algebras: *a* (*commutative*) Hopf

algebra is faithfully flat over any Hopf subalgebra, see, e.g., [28, Thm 14.1]. The proof is developed from an idea of J.C. Moore [20]. The advantage of this proof is that we don't need to assume the Hopf algebras involved to be of finite type.

Theorem 4.1.1. Let L be a flat commutative Hopf algebras over R and L' be a Hopf subalgebras. Then L is faithfully flat over L' if and only if L/L' is R-flat, i.e., L' is a special subcoalgebra of L.

Proof. Assume that L is faithfully flat over L'. Consider the tensor product of the exact sequence

$$(5) \qquad \qquad 0 \longrightarrow L' \longrightarrow L \longrightarrow Q \longrightarrow 0$$

with L over L' we get an exact sequence

$$(6) \qquad \qquad 0 \longrightarrow L \longrightarrow L \otimes_{L'} L \longrightarrow L \otimes_{L'} Q \longrightarrow 0$$

The multiplication $L \otimes_{L'} L \longrightarrow L$ splits this sequence, hence $L \otimes_{L'} Q$ is flat over L. Since L is faithfully flat over L', Q is flat over L' and therefore it is flat over R.

Conversely, assume that L/L' is R-flat. We first show that L is flat over L', i.e., for any L'-module M, $\text{Tor}_1^{L'}(M, L) = 0$. The claim holds if R is a field.

First assume that M is R-flat. Choose P_{*}, a projective resolution of L as an L'-module. Then P_{*} \otimes k is a projective resolution of L \otimes k over L' \otimes k, where k is a residue field or the fraction field of R. We have $(M \otimes k) \otimes_{(L' \otimes k)} (P_* \otimes k) \cong (M \otimes_{L'} P_*) \otimes k,$

 $H_i((M \otimes_{L'} P_*) \otimes k) \cong Tor_i^{L' \otimes k}(M \otimes k, L \otimes k)$, for all $i \ge 0$. Since $M \otimes_{L'} P_*$ is flat over R, a Dedekind ring, the universal coefficient theorem, (see, e.g., [30, Thm 3.1.6]), applies. Thus, for each $i \ge 1$, we have an exact sequence

$$0 \longrightarrow H_{i}(M \otimes_{L'} P_{*}) \otimes k \longrightarrow H_{i}((M \otimes_{L'} P_{*}) \otimes k) \longrightarrow \text{Tor}_{1}^{R}(H_{i-1}(M \otimes_{L'} P_{*}), k) \longrightarrow 0.$$

That is, for all $i \ge 1$,

$$0 \longrightarrow \text{Tor}_{i}^{L'}(M, L) \otimes k \longrightarrow \text{Tor}_{i}^{L' \otimes k}(M \otimes k, L \otimes k) \longrightarrow \text{Tor}_{1}^{R}(\text{Tor}_{i-1}^{L'}(M, L), k) \longrightarrow 0.$$

As L/L' is flat, the map $L' \otimes k \longrightarrow L \otimes k$ is injective, hence flat. Therefore $\text{Tor}_{i}^{L' \otimes k}(M \otimes k, L \otimes k) = 0$, for all $i \ge 1$. Consequently

$$\operatorname{Tor}_1^R(\operatorname{Tor}_{i-1}^{L'}(M,L),k) = \operatorname{Tor}_i^{L'}(M,L) \otimes k = 0, \text{ for all } i \geqslant 1$$

This holds for any residue field and the fraction field of R, hence $\text{Tor}_0^{L'}(M, L)$ is flat over R and $\text{Tor}_i^{L'}(M, L) = 0$ for all $i \ge 1$.

Let now M be an arbitrary L'-module. Then the R-torsion submodule M_{τ} of M is also an L'-submodule. The quotient module M/M_{τ} is then R-flat. As we have the exact sequence

$$\operatorname{Tor}_{1}^{L'}(M_{\tau}, L) \longrightarrow \operatorname{Tor}_{1}^{L'}(M, L) \longrightarrow \operatorname{Tor}_{1}^{L'}(M/M_{\tau}, L) \longrightarrow \dots$$

it suffices to show $\text{Tor}_1^{L'}(M, L) = 0$ for M being R-torsion.

For each non-zero ideal $p \subset R$, the submodule M_p of elements annihilated by p, is also an L'-submodule. As M is torsion, it is the direct limit of M_p . Since the Tor-functor commutes with direct limits, one can replace M by some M_p . Since R is a Dedekind ring, each non-zero ideal p is a product of finitely many prime ideals. Therefore each M_p has a filtration, each

grade module of which is annihilated by a certain non-zero prime ideal. Thus using induction we can reduce to the case M is annihilated by a prime ideal p. In this case $M = M \otimes k_p$ is an $L' \otimes k_p$ -module, where $k_p := R/p$ and we have

$$M \otimes_{L'} P_* = M \otimes_{L' \otimes k_p} (P_* \otimes k_p).$$

Since $P_* \otimes_R k_p$ is an $L' \otimes_R k_p$ -projective resolution of $L \otimes_R k_p$, we see that

$$\operatorname{Tor}_{\mathfrak{i}}^{L'}(M,L) = \operatorname{Tor}_{\mathfrak{i}}^{L'\otimes k_{\mathfrak{p}}}(M,L\otimes k_{\mathfrak{p}}) = 0,$$

as $L \otimes k_p$ is flat over $L' \otimes k_p$.

Finally, we show that L is faithfully flat over L'.

Let M be an L'-module, such that $M \otimes_{L'} L = 0$. Then we have

$$M_k \otimes_{L'_{L}} L_k \cong (M \otimes_{L'} L) \otimes_R k = 0$$

for any residue field k or the fraction field K of R. Since $L'_k \longrightarrow L_k$ is faithfully flat, we have $M_k = 0$ and $M_K = 0$. If M is finite over L', this implies that M = 0, according to [19, Thm 4.9]. In the general case, M always contains a non-zero finite submodule and since L is flat over L', we see that M = 0 if $M \otimes_{L'} L = 0$. Thus L is faithfully flat over L'.

As a corollary of Theorem 4.1.1 and Propositions 3.2.1, 3.2.5 we have the following theorem.

Theorem 4.1.2. Let $f : G \longrightarrow G'$ be a homomorphism of affine flat groups over R, and ω_f^o be the corresponding functor $\operatorname{Rep}^o(G') \longrightarrow \operatorname{Rep}^o(G)$.

- (i) f is faithfully flat if and only if $\omega_f^{o} : \operatorname{Rep}^{o}(G') \longrightarrow \operatorname{Rep}^{o}(G)$ is fully faithful and its image is closed under taking subobjects.
- (ii) f is a closed immersion if and only if every object of $\operatorname{Rep}^{o}(G)$ is isomorphic to a special subquotient of an object of the form $\omega_{f}(X'), X' \in \operatorname{Rep}^{o}(G')$.

Remarks 4.1.3. (i) For (commutative) Hopf algebras over a field, any injective homomorphism $L' \longrightarrow L$ is automatically faithfully flat. This is not the case for Hopf algebras over a Dedekind ring. Take for example R = k[x] and L = R[T] = k[x, T] with the R-coalgebra structure given by $\Delta(T) = T \otimes 1 + 1 \otimes T$, $\varepsilon(T) = 0$ S(T) = -T (i.e. $L = R[\mathbf{G}_{\alpha}]$), and consider the map of Hopf R-algebras

$$f: L \longrightarrow L, \quad T \mapsto xT.$$

The quotient L/f(L) is not flat over R, hence Theorem 4.1.2 implies that L is not faithfully flat over f(L).

(ii) Theorem 4.1.2 implies that, if $f : L' \longrightarrow L$ is a homomorphism of flat Hopf R-algebras, such that L/f(L') is flat over R, then L is faithfully flat over f(L').

(iii) The claim (ii) of Theorem 4.1.2 generalizes a result of dos Santos, [23, Prop. 12].

Questions 4.1.4. Theorem 4.1.2 suggests the following question: find a criterion of a Tannakian category \mathbb{C} such that its Tannakian group G is pro-algebraic in the sense that

$$G = \varprojlim_{\alpha} G_{\alpha}$$

where each group scheme G_{α} is of finite type and each structure map $G_{\beta} \longrightarrow G_{\alpha}$ is faithfully flat. Similarly, find a criterion such that G is smooth in the sense that the G_{α} above are

smooth. It seems that the local finiteness mentioned in the previous section closely relates to these problems.

4.2. Tannakian description of exact sequences. Recall that the right regular representation of G on its coordinate ring L, $(g,h) \mapsto gh : (gh)(x) = h(xg)$, $g \in G$, $h \in L$ corresponds to the right coaction of L on itself by the coproduct Δ . The left regular action of G on L, $(g,h) \mapsto gh : (gh)(x) = h(g^{-1}x)$ corresponds to the following (right) coation of L on itself:

$$a\mapsto \sum_{\mathfrak{i}} \mathfrak{a}'_{\mathfrak{i}}\otimes S(\mathfrak{a}_{\mathfrak{i}}), \quad \text{where } \Delta(\mathfrak{a})=\sum_{\mathfrak{i}} \mathfrak{a}_{\mathfrak{i}}\otimes \mathfrak{a}'_{\mathfrak{i}}, \text{ and } S \text{ denotes the antipode}$$

Let $G \longrightarrow A$ be a homomorphism of affine groups schemes over R. Let I_A be the kernel of counit $\epsilon : R[A] \longrightarrow R$, i.e. the augmentation ideal of R[A], and let $I_A R[G]$ be the ideal generated by the image of I_A in R[G]. Then the kernel of $G \longrightarrow A$ is the closed subscheme of G with coordinate ring $R[G]/I_A R[G]$. A sequence

$$1 \longrightarrow H \xrightarrow{q} G \xrightarrow{p} A \longrightarrow 1$$

is said to be exact if p is a quotient map with kernel H. We will provide a criterion for the exactness in terms of the functors

(7)
$$\operatorname{Rep}^{o}(A) \xrightarrow{p^{*}} \operatorname{Rep}^{o}(G) \xrightarrow{q^{*}} \operatorname{Rep}^{o}(H).$$

We first need a lemma relating the coordinate rings R[A], R[G] and R[H].

Lemma 4.2.1. Let $G \longrightarrow A$ be a quotient map with kernel H.

- (i) If M is a G-module, then M^H, the H-trivial submodule of M, is stable under the action of G, i.e. it is a G-submodule of M.
- (ii) R[A] is equal to $R[G]^H$ as G-modules.

Proof. (i) This follows immediately from the normality of A in G, see [17, I.3.2] for a proof.

(ii) The proof is based on the fact that R[G] is faithfully flat over R[A] and follows closely the proof for group schemes over fields, cf. [28, Sect. 15.4].

There is an isomorphism $H \times G \xrightarrow{\simeq} G \times_A G$; $(h, g) \longrightarrow (hg, g)$, which precisely means that $G \longrightarrow A$ is a principal bundle under H. In terms of the coordinate rings this isomorphism has the form

(8)
$$\varphi: \mathbb{R}[G] \otimes_{\mathbb{R}[A]} \mathbb{R}[G] \cong \mathbb{R}[H] \otimes_{\mathbb{R}} \mathbb{R}[G], \quad \mathfrak{a} \otimes \mathfrak{b} \mapsto \sum_{\mathfrak{i}} \mathfrak{q}(\mathfrak{a}_{\mathfrak{i}}) \otimes \mathfrak{a}_{\mathfrak{i}}'\mathfrak{b},$$

where $q : R[G] \longrightarrow R[H]$ denotes the quotient map and $\Delta(a) = \sum_i a_i \otimes a'_i$. The inverse map is given by

$$\mathfrak{q}(\mathfrak{a})\otimes \mathfrak{b}\mapsto \sum_{\mathfrak{i}}\mathfrak{a}_{\mathfrak{i}}\otimes S(\mathfrak{a}_{\mathfrak{i}}')\mathfrak{b}\text{,}$$

where S denotes the antipode of R[G]. One checks that this assignment depends on $q(a) \in R[H]$ but does not depend on the choice of $a \in R[G]$. Now consider the following diagram:

$$R[G]^{H} \longrightarrow R[G] \xrightarrow{d} R[G] \otimes_{R[A]} R[G]$$

$$\downarrow^{\varphi}$$

$$R[H] \otimes_{R} R[G]$$

where $d(a) := a \otimes 1 - 1 \otimes a$. Then d' is computed as follows:

$$d'(\mathfrak{a}) = \sum_{\mathfrak{i}} \mathfrak{q}(\mathfrak{a}_{\mathfrak{i}}) \otimes \mathfrak{a}'_{\mathfrak{i}} - 1 \otimes \mathfrak{a}, \quad \text{where} \quad \Delta(\mathfrak{a}) = \sum_{\mathfrak{i}} \mathfrak{a}_{\mathfrak{i}} \otimes \mathfrak{a}'_{\mathfrak{i}}.$$

Thus $R[G]^H$ is precisely the kernel of d', hence is also the kernel of d as φ is an isomorphism. On the other hand, as R[G] is faithfully flat over R[A], R[A] is the kernel of d. We conclude that $R[A] = R[G]^H$.

Theorem 4.2.2. Let us be given a sequence

$$H \xrightarrow{q} G \xrightarrow{p} A$$

with q a closed immersion and p faithfully flat. Then this sequence is exact if and only if the following conditions are fulfilled:

- (a) For an object $V \in \mathsf{Rep}^{o}(G)$, $q^{*}(V)$ in $\mathsf{Rep}^{o}(H)$ is trivial if and only if $V \cong p^{*}U$ for some $U \in \mathsf{Rep}^{o}(A)$.
- (b) Let W_0 be the maximal trivial subobject of $q^*(V)$ in $Rep^o(H)$. Then there exists $V_0 \subset V \in Rep^o(G)$, such that $q^*(V_0) \cong W_0$.
- (c) Any $W \in \mathsf{Rep}^{o}(H)$ is a quotient in (hence, by taking duals, a subobject of) $q^{*}(V)$ for some $V \in \mathsf{Rep}^{o}(G)$.

Proof. Assume that $q : H \longrightarrow G$ is the kernel of $p : G \longrightarrow A$. Then (a) and (b) follow from 4.2.1 (i) and (ii). We prove (c).

Let Ind : $\operatorname{Rep}(H) \longrightarrow \operatorname{Rep}(G)$ be the induced representation functor, it is the right adjoint functor to the restriction functor Res : $\operatorname{Rep}(G) \longrightarrow \operatorname{Rep}(H)$ that is

(9)
$$\operatorname{Hom}_{G}(V, \operatorname{Ind}(W)) \xrightarrow{\cong} \operatorname{Hom}_{H}(\operatorname{Res}(V), W).$$

One has $Ind(W) \cong (W \otimes_R R[G])^H$, where H acts on R[G] by the left regular action, [17, I.3.4]. Notice that the subspace $(W \otimes_R R[G])^H \subset W \otimes_R R[G]$ is invariant under the action of R[A], i.e. it is an R[A] submodule.

Notice that the isomorphism (8)

$$R[G] \otimes_{R[A]} R[G] \xrightarrow{\simeq} R[H] \otimes_{R} R[G]$$

is a map of G-modules where G acts on the second tensor terms by the right regular action. Since R[G] is faithfully flat over its subalgebra R[A], taking the tensor product with R[G] over R[A] commutes with taking H-invariants, hence we have

$$\mathsf{Ind}(W) \otimes_{\mathsf{R}[A]} \mathsf{R}[G] \simeq (W \otimes_{\mathsf{R}} \mathsf{R}[G])^{\mathsf{rt}} \otimes_{\mathsf{R}[A]} \mathsf{R}[G] \simeq W \otimes_{\mathsf{R}} \mathsf{R}[G].$$

1.1

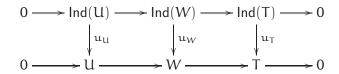
This in turns implies that the functor Ind is faithfully exact.

Setting V = Ind(W) in (9), one obtains a canonical map $u_W : Ind(W) \longrightarrow W$ in Rep(H) which gives back the isomorphism in (9) as follows:

$$\operatorname{Hom}_{G}(V, \operatorname{Ind}(W)) \ni h \mapsto \mathfrak{u}_{W} \circ h \in \operatorname{Hom}_{H}(\operatorname{Res}(V), W).$$

The map u_W is non-zero whenever W is non-zero. Indeed, since Ind is exact and faithful, Ind(W) is non-zero whenever W is non-zero. Thus if were $u_W = 0$, then (9) were the zero map for any V. On the other hand, for V = Ind(W), the right hand side contains the identity map. A contradiction, which shows that u_W cannot vanish.

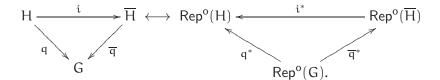
We show now that u_W is always surjective. Let $U = im(u_W)$ and $T = W/U \in \text{Rep}_f(H)$. We have the following diagram



By assumption, the composition $Ind(W) \twoheadrightarrow Ind(T) \longrightarrow T$ is 0, therefore $Ind(T) \longrightarrow T$ is a zero map, implying T = 0.

Assume now *W* is finitely generated projective over R. Then Ind(W) is torsion free. Hence Ind(W) is the union of its finitely generated R-projective modules, we can find a finitely generated G-submodule $W_0(W)$ of Ind(W) which still maps surjectively on *W*. In order to obtain the statement on the embedding of *W*, we dualize the map $W_0(W^{\vee}) \rightarrow W^{\vee}$.

Assume now that (a), (b), (c) are satisfied. Then it follows from (a) that for $U \in \mathsf{Rep}^{o}(A)$, $q^*p^*(U) \in \mathsf{Rep}^{o}(H)$ is trivial. Hence $pq : H \longrightarrow A$ is the trivial homomorphism. Recall that by assumption, q is injective, p is surjective. Let $\overline{q} : \overline{H} \longrightarrow G$ be the kernel of p. Then we have commutative diagram



It remains to show that i is surjective. We use Proposition 3.2.1. We first show the functor i^* is fully faithful. The faithfulness is obvious, we show the fullness. Let $\overline{W}_0, \overline{W}_1$ be objects in $\operatorname{Rep}^o(\overline{H})$, and $\varphi : W_0 = i^*(\overline{W}_0) \longrightarrow W_1 = i^*(\overline{W}_1)$ be a morphism of H-modules. Since \overline{H} is the kernel of p, the first part of this proof shows that there exist surjection $\overline{q}^*(V_0) \twoheadrightarrow \overline{W}_0$ and injection $\overline{W}_1 \hookrightarrow \overline{q}^*(V_1)$ where V_0, V_1 are objects of $\operatorname{Rep}^o(G)$. Thus φ combined with these maps yields a map $\widehat{\varphi} : q^*(V_0) \longrightarrow q^*(V_1)$. $\widehat{\varphi}$ corresponds to an element of $(q^*(V_1) \otimes q^*(V_0)^{\vee})^H$. By conditions (a), (b) for H and by the fact that \overline{H} also satisfies (a), (b), there exists $\psi : \overline{q}^*(V_0) \longrightarrow \overline{q}^*(V_1)$ such that

$$\widehat{\phi} = \mathfrak{i}^*(\psi) : \mathfrak{q}^*(V_0) \twoheadrightarrow \overline{W}_0 \xrightarrow{\phi} \overline{W}_1 \hookrightarrow \mathfrak{q}^*(V_1).$$

This implies that $\phi = \mathfrak{i}^*(\overline{\phi})$ for some $\overline{\phi} : \overline{W}_0 \longrightarrow \overline{W}_1$. Thus \mathfrak{i}^* is full.

For any $W \in \operatorname{Rep}^{o}(H)$, by (c) there exist V_0, V_1 in $\operatorname{Rep}^{o}(G)$ and $\varphi : q^*(V_0) \longrightarrow q^*(V_1)$ such that $W = \operatorname{im} \varphi$. Since \mathfrak{i}^* is full, $\varphi = \mathfrak{i}^* \overline{\varphi}$, hence $W \cong \mathfrak{i}^*(\operatorname{im} \overline{\varphi})$. Thus we have proved that any object in $\operatorname{Rep}^{o}(H)$ is isomorphic to the image under \mathfrak{i}^* of an object in $\operatorname{Rep}^{o}(\overline{H})$. Together with the discussion above this implies that $\overline{H} \cong H$.

5. STRATIFIED SHEAVES ON A SMOOTH SCHEME OVER A DEDEKIND RING

5.1. Stratified sheaves on smooth schemes over a Dedekind ring. Let R be a Dedekind ring, denote S := Spec(R). We shall assume that the residue fields of R are all *perfect*. Let $f : X \longrightarrow S$ be a smooth morphism with geometrically connected fibers. Consider the category str(X/S) of \mathcal{O}_X -coherent modules over the sheaf $\mathscr{D}(X/S)$ of algebras of differential operators on X/S. This is an abelian tensor category with the unit object being \mathcal{O}_X equipped with the usual Kähler differential. We call objects of str(X/S) stratified sheaves.

If R is a perfect field then objects of str(X/S) are locally free as sheaves on X (see 5.1.5). For the proof of this fact in characteristic 0 see [18] and in positive characteristic see [22].

We assume that f admits a section $\xi : S \longrightarrow X$. The pull-back along ξ provides a functor $\xi^* : str(X/S) \longrightarrow Mod(R)$. The following results are similar to those of dos Santos [23, Sect 4.2].

Proposition 5.1.1. The following claims hold:

- (i) An object of str(X/S), which is R-torsion-free, is flat (and hence is locally free) as an \mathcal{O}_X -module. Consequently, the subcategory of \mathcal{O}_X -locally free objects is closed under taking subobjects.
- (ii) The functor ξ^* : str(X/S) \longrightarrow Mod(R) is faithful and exact.

Proof. Since these are local properties, we can assume that R is a discrete valuation ring with a uniformizer t and X is an affine scheme over R, X = SpecA. Then the proof of [23, Lem. 19, Cor. 20] can be used.

(i). By the local flatness criterion, an object M of str(X/S) is flat over \mathcal{O}_X if $Tor_1^R(M, \mathcal{O}_{X_0}) = 0$ and $M_0 := M/tM$ is flat over $X_0 = X \times_{Spec(R)} Spec(R/tR)$. The second condition is trivially satisfied as X_0 is a scheme over a field R/tR. The first condition just means M is t-torsion free.

(ii) We show that ξ^* is left exact. Let m be the kernel of $\xi : A \longrightarrow R$ then ξ^* is the functor tensoring with A/m = R. Shrinking X if necessarily, we can assume the existence of a regular sequence of generators of m, say x_1, x_2, \ldots, x_n . It suffices to check that

$$\operatorname{Tor}_{1}^{A}(M, A/\mathfrak{m})$$

vanishes for any $M \in str(X/S)$. Let M_{τ} be the R-torsion part of M and $M_f := M/M_{\tau}$ then M_f is R-torsion free hence is \mathcal{O}_X -flat by (i). Thus one is led to check the claim for those M, which are R-torsion. Such an M has a filtration (which is finite as M is coherent)

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M,$$

where M_i/M_{i-1} is killed by t, so that M_i/M_{i-1} is supported on X_0 . Using the long exact sequence for Tor, one reduces the problem to the case M is supported on X_0 . Thus M is locally free as a sheaf on X_0 . Let $T_j := \text{Tor}_1^A(M, A/(x_1, ..., x_j))$. We will show by induction that $T_j = 0$

for j = 1, 2, ..., n. To see that $T_1 = 0$ we consider the exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot x_1} A \longrightarrow A/(x_1) \longrightarrow 0,$$

which shows that T_1 is the kernel of the multiplication by x_1 on M, which is 0 as M is locally free over \mathcal{O}_{X_0} . For the induction step, assuming that $T_{j-1} = 0$ and considering the short exact sequence

$$0 \longrightarrow A_{j-1} \xrightarrow{\cdot x_j} \longrightarrow A_{j-1} \longrightarrow A_j \longrightarrow 0,$$

 $A_i := A/(x_1, x_2, ..., x_j)$ we obtain a short exact sequence

$$0 \longrightarrow \mathsf{T}_{j} \longrightarrow \mathsf{A}_{j-1} \otimes_{\mathsf{A}} \mathsf{M} \xrightarrow{\cdot x_{j}} \longrightarrow \mathsf{A}_{j-1} \otimes_{\mathsf{A}} \mathsf{M}.$$

Now $A_{j-1} \otimes_A M$ is a stratified module on the smooth scheme

SpecA/
$$(t, x_1, ..., x_{j-1})$$
,

hence, as above, we conclude that $T_i = 0$.

Finally, to see that ω is faithful, it suffices to see that $\xi^*(M) = 0$ implies M = 0 for any stratified sheaf on X. We follow again the argument above: the torsion-free part is restricted to the generic fiber and the torsion part is filtered with successive quotients supported on closed fiber.

A locally free stratified sheaf is also called *stratified bundle*. Let $str^{o}(X/S)$ be the subcategory of stratified bundles. This is a rigid tensor category, and by the above proposition it is closed under taking subobjects.

Lemma 5.1.2. Let K be quotient field of R. Then the category obtained by scalar extension $R \longrightarrow K$, str^o(X/S)_K, is abelian.

Proof. It is to define quotients in $str^{o}(X/S)_{K}$. We first show the following: for $M \subset N$ in $str^{o}(X/S)$, let M^{sat} denote the saturation of M in N, i.e. the minimal extension of M in N such that N/M^{sat} is R-torsion free, then M and M^{sat} are isomorphic as objects in $str^{o}(X/S)_{K}$. Indeed, since M^{sat}/M is R-torsion, as in the proof of Proposition 5.1.1 (ii), there exists a finite filtration for M^{sat}/M such that each successive quotient is killed by a non-zero element of R. Hence M^{sat}/M is killed by a single non-zero element $a \in R$. Thus $M^{sat}/M \otimes_R K = 0$, that is, $M \otimes_R K \cong M^{sat} \otimes_R K$.

Let now $f : M \longrightarrow N$ be a morphism in $str^o(X/S)_K$, then there exists $a \in R$ such that af is a morphism in $str^o(X/S)$. The kernel of f is defined to be the kernel of af, since M is R-torsion free, the definition is independent of the choice of a. On the other hand, the co-kernel of f is defined to be the quotient of N by the saturation of Im(af). The discussion above shows that this definition is independent of the choice of a. Indeed, the saturations of Im(af) and Im(baf) in N are the same for any $b \in R$.

Based on results of section 2 we make the following definition.

Definition 5.1.3. The fiber functor ξ^* makes str^o(X/R) a Tannakian lattice, its Tannakian group, denoted by $\pi(X/S, \xi)$, is called the relative fundamental group scheme of X/S.

Let $M \in str^{o}(X/S)$. Consider the full subcategory $\langle M \rangle_{s}^{\otimes}$ of $str^{o}(X/S)$, consisting of special subquotients of direct sums of tensors powers of the form

$$\mathsf{T}^{\mathfrak{a},\mathfrak{b}}(\mathsf{M}) := \mathsf{M}^{\otimes \mathfrak{a}} \otimes \mathsf{M}^{\vee \otimes \mathfrak{b}}, \quad \mathfrak{a},\mathfrak{b} \in \mathbb{N}.$$

Then $\langle M \rangle_s^{\otimes}$ is also a Tannakian lattice. Its Tannakian group scheme G(M) is called the differential Galois group scheme of M. This group was first studied in [23].

5.1.4. We don't know if str(X/S) is a Tannakian category over R, be cause we cannot check if any stratified sheaf is representable as a quotient of a stratified bundle. However we can define the abelian envelope of $str^{o}(X/S)$, denoted by $\mathcal{C} = \mathcal{C}(X/S)$, as the full subcategory of str(X/S)consisting of stratified sheaves which can be represented as quotients of stratified bundles. According to Proposition 5.1.1, \mathcal{C} is a (neutral) Tannakian category over R. We conclude that \mathcal{C} is equivalent to the representation category of $\pi(X/S, \xi)$ by means of the fiber functor ξ^* . Thus \mathcal{C} is the abelian envelope of $str^{o}(X/S)$ in the sense of 2.4.1.

5.1.5. Let s be a closed point of S, $k := k_s = R/p_s$ – the residue field of s, and let X_s denote the fiber of f at s. Consider the category $str(X_s/k)$. Its objects are automatically locally free as \mathcal{O}_{X_s} -modules, cf. 5.1. Thus $str(X_s/k)$ is an abelian rigid tensor category over k.

The fiber of str(X/S) at $s \in S$ is defined as in Remark 1.3.4, and is denoted by $str(X/S)_s$. This full subcategory of str(X/S) is identified with the category of stratified bundle on X_s . Indeed, the restriction functor $str(X/S) \longrightarrow str(X_s/k)$ (given by pulling-back along the closed immersion $X_s \longrightarrow X$) can be identified with the functor which associates to each object M of str(X/S) the quotient M/p_sM . In particular, $str(X_s/k)$ is naturally a subcategory of str(X/S), consisting of those objects which are annihilated by p_s .

Lemma 5.1.6. $C(X/S)_s$ is a full subcategory of $str(X_s/k)$, closed under taking subquotients.

Proof. This is obvious by definition (see 1.3.4). A stratified sheaf M_0 is an object of $\mathcal{C}(X/S)_s$ if it is annihilated by p_s and can be presented as a quotient of a stratified bundle, say M. Thus any quotient of M_0 is again in $\mathcal{C}(X/S)_s$. On the other hand, if N_0 is a subobject of M_0 then taking the pull-back of N_0 along the projection map $M \longrightarrow M_0$ we get a subobject N of M which surjects onto N_0 . Since N itself is locally free, we conclude that N_0 is in $\mathcal{C}(X/S)_s$.

Remarks 5.1.7. In general we don't know if M can be presented in the from $M_0 = M/\pi M$, that is if M_0 is the fiber at s of some stratified bundle M. The difference between C(X/S) and str(X/S) consists essentially in those stratified sheaves supported in X_s and not representable as a quotient of a stratified bundle. See 5.2 below.

5.1.8. In the rest of this subsection we shall assume that R contains a field k which is mapped isomorphically onto any residue field. Thus R can be see as the coordinated ring on the affine curve S on k (recall that k is assumed to be perfect).

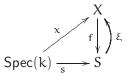
Consider X as a scheme over k. A stratified sheaf on X/k is automatically locally free on X and is called an *absolute* stratified bundle. There is a natural "inflation" functor from str(X/k) to $str^{o}(X/S)$: consider an (absolute) stratified bundle on X/k as a (relative) stratified bundle on X/S. In the other direction, pulling-back along f yields the functor $\omega_{f} = f^{*} : str(S/k) \rightarrow Stratified bundle other direction and the strategies of the strate$

str(X/k). Thus, to summarize, we have the following sequence of tensor functors:

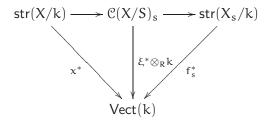
$$\operatorname{str}(S/k) \xrightarrow{\omega_{f}} \operatorname{str}(X/k) \xrightarrow{\operatorname{infl}} \mathcal{C}(X/S) \xrightarrow{\operatorname{res}} \mathcal{C}(X/S)_{s} \longrightarrow \operatorname{str}(X_{s}/k)$$

Let x be a k-rational point of X/k. The fiber functor at x makes $\operatorname{str}(X/k)$ a neutral Tannakian category. Its Tannakian group is denoted by $\pi(X/k, x)$ and called the fundamental group scheme of X at x. Let s = f(x). The functor ω_f is compatible with the fiber functors at s and x. Thus, according to A.1.11, we have a homomorphism of fundamental group schemes $f_* : \pi(X/k, x) \longrightarrow \pi(S/k, s)$. The restriction functor $\operatorname{str}(X/k) \longrightarrow \operatorname{str}(X_s/k)$ is also a tensor functor and is compatible with the fiber functors at x, hence induces a homomorphism $\pi(X_s/k, x) \longrightarrow \pi(X/k, x)$.

Assume now that $x = \xi(s)$:



Then the fiber functors x^* , ξ^* and f_s^* are compatible in the sense that the following diagram is commutative:



Hence it yields a sequence of group scheme homomorphism

$$\pi(X_s/k, x) \longrightarrow \pi(X/S, \xi)_s \longrightarrow \pi(X/k, x) \longrightarrow \pi(S/k, s).$$

Proposition 5.1.9. The homomorphism $\pi(X_s/k, x) \longrightarrow \pi(X/S, \xi)_s$ is surjective.

Proof. Here, the group schemes are defined over a field. Hence we can use the criterion for surjectivity of Deligne-Milne [5, Thm. 2.21]. We show that for each object $M \in \text{str}(X/S)_s$, when considered as object in $\text{str}(X_s/k, x)$ all its subobjects will be an object in $\text{str}(X/S)_s$. This is obvious from the fact that the category $\text{str}^o(X/S)$ is closed under taking subobjects. There exists by assumption an $X \in \text{str}^o(X/S)$ which surjects on M. We have the following pull-back diagram



Since Y is a subobject of X, it is itself locally free.

5.1.10. In [24] dos Santos proved that the following homotopy sequence is exact:

(10)
$$\pi(X_s, x) \longrightarrow \pi(X, x) \longrightarrow \pi(S, s) \longrightarrow 1,$$

provided that f is a proper map. Hence in this case we also have an exact sequence

$$\pi(X/S,\xi)_s \longrightarrow \pi(X/k,x) \longrightarrow \pi(S/k,s) \longrightarrow 1.$$

The general Tannakian duality applied to ξ^* and the categories str(X/k) and str(S/k) yields the fundamental groupoid schemes $\Pi(X/k, \xi)$ and $\Pi(S/k, \xi)$, and the functor $\omega_f : str(S/k) \longrightarrow$ str(X/k) yields a surjective homomorphism $f_* : \Pi(X, \xi) \longrightarrow \Pi(S, \xi)$. The kernel of this groupoid homomorphism, is by definition

$$L := S \times_{\Pi(S,\xi)} \Pi(X,\xi),$$

where the map $S \longrightarrow \Pi(X, \xi)$ is given by the unit element. This is a flat group scheme over S. On other hand, the inflation functor $str(X/k) \longrightarrow str(X/S)$ induces a homomorphism $\pi(X/S, \xi) \longrightarrow \Pi(X/k, \xi)$. The following question is motivated by dos Santos' result mentioned above.

Questions 5.1.11. Assume that $f : X \longrightarrow S$ is a smooth, proper map with connected fibers. Is the following sequence exact

$$\pi(X/S,\xi) \longrightarrow \Pi(X/k,\xi) \longrightarrow \Pi(S/k,\xi) \longrightarrow 1?$$

5.2. The case of a complete discrete valuation ring. Let A = k[[t]] where k is a perfect field, with quotient field K = k((t)). Let \mathfrak{X} be a smooth connected formal affine scheme over SpfA. Let X_0 be the special fiber of \mathfrak{X} and let X be the generic fiber. Assume that \mathfrak{X} admits an A-rational point ξ . Our aim is to show that the category str(\mathfrak{X}/A) is Tannakian.

We will show that str^o(\mathfrak{X}/A) is subcategory of definition in str(\mathfrak{X}/A), that is, any stratified sheaf in str(\mathfrak{X}/A) is a quotient of a stratified bundle. First we need the following.

Proposition 5.2.1. The restriction functor from $str(\mathfrak{X}/k)$ to $str(X_0/k)$ is an equivalence. In particular, any exact sequence in $str(X_0/k)$ can be lifted to an exact sequence in $str(\mathfrak{X}/k)$.

Proof. If k is of positive characteristic, this is a result of Gieseker [13, Lemma 1.5]. He constructed an explicit lift of a stratified bundle on X_0/k to a stratified bundle on \mathfrak{X}/k and showed that this lift yields a functor which is quasi-inverse to the restriction functor, thus giving the equivalence.

The case of zero characteristic can be proved using the method of Katz in the proof of [18, Prop. 8.8]. Let M be a stratified module over $\mathscr{D}(\mathfrak{X}/k)$. The action of this algebra on M will be denoted as usual by ∇ . Since $\mathcal{O}_{\mathfrak{X}}$ contains the field k, $\mathscr{D}(\mathfrak{X}/k)$ is generated by the derivations, that is a stratification is nothing but a flat connection. We first show that it is locally free. By means of Proposition 5.1.1 it suffices to show that M is t-torsion free. This is a local property on \mathfrak{X} .

Let (x_1, \ldots, x_r, t) be local coordinates on \mathfrak{X} . Thus ∂_{x_i} 's commute each other and commute with ∂_t , and we have

$$\partial_{x_i}(x_j) = \delta_{ij}; \quad \partial_{x_i}(t) = \partial_t(x_i) = 0; \quad \partial_t(t) = 1.$$

One considers the k-linear operator $P := \sum_{i=0}^{\infty} \frac{(-t)^i}{i!} \partial_t^i$. It has the following properties, cf. [18,

Sect. 8]:

$$\mathsf{P}^2=\mathsf{P};\quad\mathsf{P}(\mathsf{m})=\mathsf{m}\;(\text{mod }\mathsf{t}\mathsf{M});\quad\mathsf{P}(\mathsf{f}\mathsf{m})=\mathsf{f}(\mathsf{0})\mathsf{P}(\mathsf{m}),\quad\mathsf{f}\in\mathsf{A},\mathsf{m}\in\mathsf{M}.$$

Hence, setting $M^{\nabla_t} := \text{ker} \nabla(\partial_t)$, we have

(11)
$$M^{\nabla_t} = imP; \quad M = M^{\nabla_t} \oplus tM; \quad M^{\nabla_t} \cong M_0 := M/tM.$$

Further the map $A \otimes M^{\nabla_t} \longrightarrow M$, $f \otimes \mathfrak{m} \mapsto \mathfrak{f}\mathfrak{m}$ is injective, in particular, M^{∇_t} is t-torsion free.

Assume that tm = 0 for some $m \in M$. If $m \neq 0$ it has a unique presentation $m = t^{k-1}(tm_1 + tm_2)$ \mathfrak{m}_0), where k > 0 maximal, $\mathfrak{m}_0 \in \mathcal{M}^{\nabla_t}$ (this is due to the completeness of the t-adic topology on $\mathcal{O}_{\mathfrak{X}}$). Then we have

$$0 = \nabla(\mathfrak{d}_{t})^{k}(\mathfrak{tm}) = \nabla(\mathfrak{d}_{t})^{k}(\mathfrak{t}^{k+1}\mathfrak{m}_{1}) + k!\mathfrak{m}_{0},$$

(since $\nabla(\mathfrak{d}_t)(\mathfrak{m}) = 0$). Hence $\mathfrak{m}_0 \in tM$, which implies $\mathfrak{m}_0 = 0$, contradiction. Hence $\mathfrak{m} = 0$. Thus M is t-torsion free, hence locally free over $\mathcal{O}_{\mathfrak{X}}$ and consequently the restriction functor $M \mapsto M_0 = M/tM$ is exact. It also implies that str (\mathfrak{X}/k) is an abelian rigid tensor category over k.

A section of M (as an object of $str(\mathfrak{X}/k)$) is horizontal iff (locally) it lies in $M^{\nabla_t} \cong M_0$ and is annihilated by $\nabla(\partial_{x_i})$, and hence iff its image in M_0 is a horizontal section of M_0 as an object of str(X_0/k). We conclude that the restriction functor str(\mathfrak{X}/k) \longrightarrow str(X_0/k) is faithful.

Conversely, the third isomorphism in (11) shows that on an open U of X_0 (which topologically homeomorphic to X), small enough so that local coordinates on it exist, a horizontal section of $M_0|_U$ can be uniquely lifted to a horizontal section of $M|_U$. Let now s_0 be a horizontal section of $M_0 \in str(X_0/k)$. Consider an open covering (U_α) of X_0 such that on each U_α there exist local coordinates (x_i, t) . Let $s_{0,\alpha}$ be the restriction of s_0 on U_{α} . Lift $s_{0,\alpha}$ to a horizontal section s_{α} of M on U_{α} . The restrictions of s_{α} and s_{β} on $U_{\alpha\beta}$ agree as they are liftings of the same section. Hence the s_{α} 's glue together to give a horizontal section of M on X. Thus the restriction functor is full.

It remains to show that this functor is essentially surjective, that is, each stratified bundle M_0 on X_0/k can be lifted to a stratified bundle on \mathfrak{X}/k . We first assume that on X there exist global coordinates (x_1, \ldots, x_n, t) and that M_0 is free over \mathcal{O}_{X_0} with basis (e_i^0) . Given this, we will show that a flat connection on M_0 can be lifted to $M = \langle e_i \rangle_{\mathcal{O}_{\mathfrak{X}}}$.

Consider the action of the operator P defined above on the algebra $\mathcal{O}_{\mathfrak{X}}$. We have $(D := \nabla(\partial_t))$

$$\begin{split} \mathsf{P}(\mathfrak{a}\mathfrak{b}) &= \sum_{i} \frac{(-t)^{i}}{i!} \mathsf{D}^{i}(\mathfrak{a}\mathfrak{b}) \\ &= \sum_{i} \frac{(-t)^{i}}{i!} \sum_{j} \binom{i}{j} \mathsf{D}^{j}(\mathfrak{a}) \mathsf{D}^{i-j}(\mathfrak{b}) \\ &= \sum_{j} \frac{(-t)^{j}}{j!} \mathsf{D}^{j}(\mathfrak{a}) \sum_{i \ge j} \frac{(-t)^{i-j}}{i!} \mathsf{D}^{i-j}(\mathfrak{b}) \\ &= \mathsf{P}(\mathfrak{a}) \mathsf{P}(\mathfrak{b}). \end{split}$$

Hence the isomorphism $\phi : \mathcal{O}_{\mathfrak{X}^{t}}^{\nabla_{t}} \longrightarrow \mathcal{O}_{X_{0}}$, induced by P, is an isomorphism of algebras. Notice that, since $[\partial_{\chi_{t}}, \partial_{t}] = 0$ we have $[\partial_{\chi_{t}}, D] = 0$, for all i. Thus ϕ commutes with the action of $\partial_{\chi_{t}}$.

Let ψ be the inverse of φ . Assume that the actions of $\nabla(\partial_{x_i})$ on the basis (e_j^0) is given by a set of matrices $a_i = (a_{ij}^k)$:

$$\nabla(\mathfrak{d}_{x_{\mathfrak{i}}})(e_{\mathfrak{j}}^{0})=\sum_{k}\mathfrak{a}_{\mathfrak{i}\mathfrak{j}}^{k}e_{k}^{0}.$$

The flatness of ∇ is expressed in terms of the Maurer-Cartan equation involving the matices (a_{ij}^k) and their partial derivatives in x_i 's. This equation is preserved by ψ , which means we can lift them to a set of matrices $A_j = (A_{ij}^k)$ such that the equations:

$$\nabla(\mathfrak{d}_{\mathbf{x}_{i}})(e_{j}) = \sum_{k} A_{ij}^{k} e_{k}$$

defines a flat connection on \mathfrak{X}/A .

Finally we simply set $\nabla(\partial_t)(e_i) = 0$. It is straightforward to check that $\nabla(\partial_t)$ commutes with $\nabla(\partial_{x_i})$ using the fact that $\nabla(\partial_t)(A_{ij}^k) = 0$ as these elements lie in A^{∇_t} . Thus, we have constructed a flat connection on M.

In the general case we consider an open covering of X_0 , such that on each open, the connection M_0 is free and local coordinates exist. Then on each open we can lift M_0 . As the lift on each open is unique, they glue together to give a lift of M_0 on the whole \mathfrak{X} .

Notice that if a stratified sheaf E_0 on \mathfrak{X}/A is annihilated by t then it can be considered as a sheaf on X_0/k and hence can be lifted to a stratified bundle on \mathfrak{X}/k , say E and we have an exact sequence

(12)
$$0 \longrightarrow \mathsf{E} \xrightarrow{[\mathsf{t}]} \mathsf{E} \longrightarrow \mathsf{E}_0 \longrightarrow 0$$

where [t] denotes the map multiplying by t. Thus in this case E_0 is a quotient of E (considered as stratified sheaf on \mathfrak{X}/A).

Proposition 5.2.2. Each object of $str(\mathfrak{X}/A)$ is a quotient of an object of $str(\mathfrak{X}/A)^0$. Consequently, $str(\mathfrak{X}/A)$ is a Tannakian category.

Proof. Let E be an object of $str(\mathfrak{X}/A)$. Then the subsheaf E_t consisting of sections annihilated by some power of t is invariant under the stratification. We have an exact sequence

(13)
$$0 \longrightarrow \mathsf{E}_{\mathsf{t}} \longrightarrow \mathsf{E} \longrightarrow \mathsf{E}_{\mathsf{fr}} \longrightarrow 0,$$

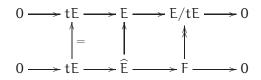
with E_{fr} a t-torsion free stratified sheaf (hence is locally free by the lemma above). There exists a least integer r such that E_t is annihilated by t^r . We will use induction on r.

For r = 1, the subsheaf $tE \subset E$ is t-torsion free. Indeed, if a section ts in tE is torsion then s is itself torsion, hence is annihilated by t. Consider the exact sequence

$$0 \longrightarrow tE \longrightarrow E \longrightarrow E/tE \longrightarrow 0.$$

The sheaf E/tE is in str(X₀/k), hence can be lifted to a stratified bundle F on \mathfrak{X}/k : F \longrightarrow E/tE. Pull back the above sequence along this map (considered as morphism in str(\mathfrak{X}/A)), we get the

following commutative diagram



In particular, the map $\widehat{E} \longrightarrow E$ is surjective. But \widehat{E} is torsion free as the two sheaves tE and F are locally free. Thus \widehat{E} is the needed stratified bundle on \mathfrak{X}/A .

Let now E be such that E_t , the subsheaf of torsion sections, is annihilated by t^n , n > 1. Let E_0 be the subsheaf of E of section annihilated by t. Then we have exact sequence

$$0 \longrightarrow E_0 \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

where for E' its torsion part E'_t is annihilated by t^{n-1} . By induction we can lift E' to F' and hence, by pulling-back, we can lift E:

$$0 \longrightarrow E_0 \longrightarrow E \longrightarrow E' \longrightarrow 0$$

$$\uparrow = \qquad \uparrow \qquad \uparrow$$

$$0 \longrightarrow E_0 \longrightarrow F \longrightarrow F' \longrightarrow 0$$

Since F' is locally free, E_0 is the torsion subsheaf of F and we can lift F.

Remarks 5.2.3. Y. Andre has given in [1, 3.2.1.5] an example showing that the differential Galois group of a connection in str^o(\mathfrak{X}/A) may be of infinite type over A.

APPENDIX A. TANNAKIAN DUALITY FOR FLAT COALGEBRAS OVER DEDEKIND RINGS

In this appendix we give a quick, complete and self-contained proof of Theorem 1.2.2. First we will recall the notion ind-category of an abelian category. The two equivalent descriptions of the ind-category will play a crucial role in Saavedra's proof. A category \mathcal{I} is called a filtered category if to every pair i, j of objects in \mathcal{I} there exists an object k such that Hom(i, k) and Hom(j, k) are both not empty, and for every pair $u, v : i \longrightarrow j$, there exists a morphism $w : j \longrightarrow k$ such that wu = wv.

Definition A.1.1. *Ind-categories.* Let \mathcal{C} be an abelian category. The category $Ind(\mathcal{C})$ consists of functors $X : \mathcal{I} \longrightarrow \mathcal{C}$, where \mathcal{I} is a filtering category. We usually denote X_i for $X(i), i \in \mathcal{I}$, an write

$$X = \lim_{i \in \mathcal{I}} X_i.$$

For two objects $X = \varinjlim_{i \in \mathcal{I}} X_i$ and $Y = \varinjlim_{j \in \mathcal{J}} Y_j$ their hom-set is defined to be

$$\operatorname{Hom}(X,Y) := \varprojlim_{i \in \mathcal{I}} \varinjlim_{j \in \mathcal{J}} \operatorname{Hom}(X_i,Y_j).$$

Let $\omega : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. The extension of ω , $\mathsf{Ind}(\omega) : \mathsf{Ind}(\mathcal{C}) \longrightarrow \mathsf{Ind}(\mathcal{D})$ is defined by

$$\operatorname{Ind}(\omega)(\varinjlim_{i} X_{i}) := \varinjlim_{i} \omega(X_{i}).$$

There is an alternative description of $Ind(\mathcal{C})$. Denote $Lex(\mathcal{C}^{op}, Sets)$ the category of left exact functors from \mathcal{C}^{op} to the category of sets. For $X = \lim_{i \to \infty} X_i$ we define functor

$$\varinjlim_{i} h_{X_{i}}(-) := \varinjlim_{i} \operatorname{Hom}(-, X_{i}) \in \operatorname{Lex}(\mathcal{C}^{\operatorname{op}}, \operatorname{Sets}).$$

This yields a functor $Ind(\mathcal{C}) \longrightarrow Lex(\mathcal{C}^{op}, Sets)$ which is an equivalence (cf.[2], I.8.3.3). Recall that the Hom-sets for objects of $Lex(\mathcal{C}^{op}, Sets)$ are by definition the sets of natural transformations. For simplicity, we shall use the notation Hom(F, G) instead of Nat(F, G) for objects of this category.

A.1.2. Suppose that C is an R-linear *Noetherian abelian* category. Let $Lex_R(C^{op}, Mod(R))$ be category of R-linear left exact functors from C^{op} to the category of modules Mod(R). Then the natural functor

$$\operatorname{Lex}_{R}(\mathcal{C}^{\operatorname{op}}, \operatorname{Mod}(R)) \xrightarrow{\simeq} \operatorname{Lex}(\mathcal{C}^{\operatorname{op}}, \operatorname{Sets})$$

is an equivalence (cf. Gabriel [12, II]). Thus, for an R-linear Noetherian abelian category we have an equivalence

$$\mathsf{Ind}(\mathfrak{C})\simeq\mathsf{Lex}_{\mathsf{R}}(\mathfrak{C}^{op},\mathsf{Mod}(\mathsf{R})),\quad X=\varinjlim_{\mathfrak{i}}X_{\mathfrak{i}}\longmapsto\varinjlim_{\mathfrak{i}}h_{X_{\mathfrak{i}}}(-).$$

Further the category $Ind(\mathcal{C})$ is locally Noetherian and the inclusion $\mathcal{C} \longrightarrow Ind(\mathcal{C})$ identifies \mathcal{C} with the full subcategory of Noetherian objects in $Ind(\mathcal{C})$, [12, II,4, Thm.1].

The following are our main examples.

Example A.1.3. The category $Mod_f(R)$ of finitely generated R-modules, where R is a Noetherian ring, is a Noetherian category. Its Ind category is precisely the category Mod(R) of all R-modules. This is obvious.

Example A.1.4. Let L be a coalgebra over a commutative ring R. Denote by Comod(L) the category of right L-comodules and by $Comod_f(L)$ the subcategory of comodules which are finitely generated as R-module. Then:

- (i) If L is flat over R then Comod(L) is an abelian category. In fact, the flatness of L implies that the kernel of a homomorphism of L-comodules is equipped with a natural coaction of L. In particular, the forgetful functor from Comod(L) to Mod(R) is exact. The converse is also true: if the forgetful functor preserves kernels then L is flat over R.
- (ii) Assume that L is flat over R and R is Noetherian. According to Serre [25, Cor. 2] each L-comodule is the union of its R-finite subcomodules. Consequently, Comod(L) is locally Noetherian and $Comod_f(L)$ is the full subcategory of Noetherian objects.

Let \mathcal{C} be an R-linear abelian category, and $\omega : \mathcal{C} \longrightarrow \mathsf{Mod}_f(\mathsf{R})$ be an R-linear exact faithful functor. Suppose that there exists a full subcategory of definition \mathcal{C}° in \mathcal{C} . Our aim is to show that there exists a flat R-coalgebra L such that ω induces an equivalence between $\mathsf{Comod}_f(\mathsf{L})$ and \mathcal{C} , and between $\mathsf{Comod}(\mathsf{L})$ and $\mathsf{Ind}(\mathcal{C})$.

The functor ω induces a functor $Ind(\mathcal{C}) \longrightarrow Mod(R)$, which we, by abuse of language, will denote simply by ω . Recall that we identify $Ind(\mathcal{C})$ with $Lex(\mathcal{C}^{op}, Mod(R))$, the category of left exact functors on \mathcal{C}^{op} with values in Mod(R). The key technique is to use alternatively these two equivalent descriptions of one category.

A.1.5. For any R-algebra A, we define functor

 $\mathsf{F}^{\mathsf{A}}: \mathfrak{C}^{op} \longrightarrow \mathsf{Mod}(\mathsf{A}), \quad \mathsf{X} \longmapsto \mathsf{Hom}(\omega(\mathsf{X}),\mathsf{A}).$

Then F^A is an object of Lex(\mathcal{C}^{op} , Mod(R)). Set $F := F^R$. There is a natural A-linear transformation $A \otimes F \longrightarrow F^A$:

$$\theta_X : A \otimes \operatorname{Hom}(\omega(X), \mathbb{R}) \longrightarrow \operatorname{Hom}(\omega(X), A), \quad a \otimes f \longmapsto af.$$

Lemma A.1.6. The A-linear transformation $\theta : A \otimes F \longrightarrow F^A$ given above is an isomorphism.

Proof. For any $K, G \in Lex(\mathbb{C}^{op}, Mod(R))$ we denote K^o, G^o their restrictions to $(\mathbb{C}^o)^{op}$, respectively. We claim that

(14)
$$\operatorname{Hom}(K, G) \simeq \operatorname{Hom}(K^{\circ}, G^{\circ}).$$

Indeed, let $\theta \in \text{Hom}(K^o, G^o)$, that is we have a family $\theta_X : K^o(X) \longrightarrow G^o$ for $X \in C^o$ commuting with morphism in C^o . Since each object of C can be represented as a cokernel of a morphism $X_1 \longrightarrow X_2$ in C^o , we see that θ extends uniquely to a natural transformation $K \longrightarrow G$ (as these functors are left exact on C^{op}).

For $X \in \mathfrak{C}^{o}$, $\omega(X)$ is finite projective over R, hence

$$F^{A}(X) = Hom(\omega(X), A) \simeq Hom(\omega(X), R) \otimes A = A \otimes F(X).$$

Therefore, for any $G \in Lex(\mathcal{C}^{op}, Mod(R))$, we have

(15)
$$\operatorname{Hom}((F^{A})^{o}, G^{o}) = \operatorname{Hom}((A \otimes F)^{o}, G^{o})$$

and (14) yields

(16)
$$\operatorname{Hom}(F^{A}, G) = \operatorname{Hom}(A \otimes F, G).$$

So we have $F^A \simeq A \otimes F$.

We will show that $L := \omega(F)$ is the coalgebra to be found. To show this, first we will need

Lemma A.1.7. For any $X \in Lex(C^{op}, Mod(R))$ and R-algebra A we have the following A-linear isomorphism:

(17)
$$\operatorname{Hom}(X, F^{A}) \simeq \operatorname{Hom}_{A}(A \otimes \omega(X), A) = \operatorname{Hom}_{R}(\omega(X), A).$$

Proof. Every $X \in Lex(\mathcal{C}^{op}, Mod(R))$ can be represented as $X = \varinjlim_{i} h_{X_i}(X_i \in \mathcal{C})$, where h_{X_i} is a functor over \mathcal{C} , defined by $h_{X_i}(-) := Hom_{\mathcal{C}}(-, X_i)$. Hence we have

It is easy to see that all isomorphisms are A-linear.

Isomorphism (17) for A = R and X = F reads $Hom(F, F) \simeq Hom_R(\omega(F), R)$. We denote $L := \omega(F)$ and let $\varepsilon : L \longrightarrow R$ be the map on the right hand side that corresponds to the identity transformation on the left hand side of this isomorphism. The next lemma shows that one can replace the algebra A in (17) by any R-module M to get R-linear isomorphisms.

Lemma A.1.8. There exists a natural R-linear isomorphism extending (17)

(18) $\Phi_{X,M}: \operatorname{Hom}(X, M \otimes F) \simeq \operatorname{Hom}_{R}(\omega(X), M),$

which is given explicitly by

 $\Phi_{X,M}(f) = (\mathsf{id}_M \otimes \epsilon) \circ \omega(f).$

Proof. For any R-module M, we can make $R \oplus M$ into an R-algebra by letting M be an ideal with square null. Hence the isomorphism (18) is a direct consequence of (17). By definition $\Phi_{F,R}$ is given by

$$\Phi_{\mathsf{F},\mathsf{R}}(\mathsf{f}) = \varepsilon \circ \omega(\mathsf{f}).$$

Each R-linear map $\iota: R \longrightarrow M$ induces by functoriality the commutative diagram

$$\begin{array}{c} \operatorname{Hom}(F,F) \xrightarrow{\varepsilon \circ \omega(-)} \operatorname{Hom}_{R}(\omega(F),R) \\ (\iota \otimes \operatorname{id}_{F}) \circ - \downarrow & \downarrow \iota \circ - \\ \operatorname{Hom}(F,M \otimes F) \xrightarrow{\Phi_{F,M}} \operatorname{Hom}_{A}(\omega(F),M) \end{array}$$

Now, the identity on F yields the equality:

$$\iota \circ \epsilon = \Phi_{F,M}(\iota \otimes \mathsf{id}_{\omega(F)}) : \omega(F) \longrightarrow M.$$

Hence, for $m = \iota(1)$, we have $\Phi_{F,M}(l) = \varepsilon(l)m$, $l \in \omega(F)$. Thus the claim holds for X = F. Since the ω and Hom-functor in the first variant commute with direct limits we conclude that the claim hold of $X = N \otimes F$ for any R-module N. Now the general case follows from the following diagram

$$\operatorname{Hom}(M \otimes F, M \otimes F) \xrightarrow{\Phi_{F,M \otimes F}} \operatorname{Hom}(M \otimes \omega(F), M) \xrightarrow[(-)\circ \omega(f)]{} \operatorname{Hom}(X, M \otimes F) \xrightarrow{\Phi_{X,M}} \operatorname{Hom}(\omega(X), M)$$

applied for the identity of $M \otimes F$:

$$\Phi_{X,M}(f) = \Phi_{F,M}(\mathsf{id}) \circ \omega(f) = (\mathsf{id}_M \otimes \varepsilon) \circ \omega(f).$$

Proposition A.1.9. Let $L := \omega(F)$. Then it is a coalgebra with ε being the counit and ω factors though a functor

$$\operatorname{Ind}(\mathfrak{C}) \longrightarrow \operatorname{Comod}(L).$$

Proof. Choose $M = \omega(X)$ in (18) we have a morphism $\sigma_X : X \longrightarrow \omega(X) \otimes F$ which corresponds to the identity element $id_{\omega(X)}$ under the isomorphism $\Phi_{X,\omega(X)}$ of Lemma A.1.8, thus we have

(19)
$$(\mathrm{id}_{\omega(X)}\otimes \varepsilon)\circ \omega(\sigma_X)=\mathrm{id}_{\omega(X)}.$$

 \square

For any morphism $\lambda : X \longrightarrow Y$ in $Ind(\mathcal{C})$, according to A.1.8 we have the following equalities:

$$\Phi_{X,\omega(Y)} \left((\omega(\lambda) \otimes \mathsf{id}_{\mathsf{F}}) \circ \sigma_X \right) = \omega(\lambda),$$

$$\Phi_{X,\omega(Y)} \left(\sigma_Y \circ \lambda \right) = \omega(\lambda).$$

Thus $(\omega(\lambda) \otimes id_F) \circ \sigma_X = \sigma_Y \circ \lambda$, i.e, the following diagram commutes:

$$\omega(X) \otimes F \xrightarrow{\omega(\lambda) \otimes \mathrm{id}_F} \omega(Y) \otimes F.$$

For $Y = \omega(X) \otimes F$ and $\lambda = \sigma_X$, we get

Applying ω on this diagram we obtain a commutative diagram in Mod(R):

(22)
$$\begin{array}{c} \omega(X) \xrightarrow{\omega(\sigma_X)} \omega(X) \otimes L \\ \downarrow^{\omega(\sigma_X)} \downarrow & \downarrow^{id \otimes \Delta} \\ \omega(X) \otimes L \xrightarrow{\omega(\sigma_X) \otimes id_L} \omega(X) \otimes L \otimes L \end{array}$$

where $\Delta := \omega(\sigma_F)$. Together with (19), this diagram for X = F gives a coalgebra structure on L with Δ being the coproduct and hence, for any X, it gives a comodule structure of L on $\omega(X)$. \Box

Proof. (of Theorem 1.2.2) Let L be defined as in Proposition A.1.9. We consider ω as a functor $\mathcal{C} \longrightarrow \text{Comod}_f(L)$. It is to show that ω is an equivalence of category. By definition it is faithful. To see the fullness, suppose X, $Y \in \mathcal{C}$ and $\alpha : \omega(X) \longrightarrow \omega(Y)$ is a homomorphism of L-comodules, i.e., we have

$$(\alpha \otimes \mathsf{id}) \circ \omega(\sigma_X) = \omega(\sigma_Y) \circ \alpha : \omega(X) \longrightarrow \omega(Y) \otimes L$$

Then $\omega(X) \xrightarrow{\alpha} \omega(Y) \xrightarrow{\omega(\sigma_Y)} \omega(Y) \otimes L$ is the image under ω of the morphism

$$X \xrightarrow{\sigma_X} \omega(X) \otimes F \xrightarrow{\alpha \otimes \mathsf{id}_F} \omega(Y) \otimes F.$$

Notice that (22) (for X replaced by Y) yields a split exact sequence

(23)
$$0 \longrightarrow \omega(Y) \xrightarrow{\omega(\sigma_Y)} \omega(Y) \otimes L \xrightarrow{\delta} \omega(Y) \otimes L \otimes L,$$

where the second homomorphism is $\delta = id \otimes \Delta - \omega(\sigma_X) \otimes id$, and the splitting is given by $id \otimes \varepsilon : \omega(Y) \otimes L \longrightarrow \omega(Y)$. This sequence is the similar image under ω of the sequence coming from (21):

$$0 \longrightarrow Y \longrightarrow \omega(Y) \otimes F \longrightarrow \omega(Y) \otimes L \otimes F.$$

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Hence the latter sequence is also exact. On the other hand, it follows from the faithfulness of ω that the composed map

$$X \xrightarrow{\sigma_X} \omega(X) \otimes F \xrightarrow{\alpha \otimes id} \omega(Y) \otimes F \longrightarrow \omega(Y) \otimes L \otimes F$$

is the zero morphism (since its image under ω is zero by means of (22) and the fact that α is a homomorphism of L-comodules). Consequently, the morphism $X \xrightarrow{\sigma_X} \omega(X) \otimes F \xrightarrow{\alpha \otimes id} \omega(Y) \otimes F$ factor through a morphism $f: X \longrightarrow Y$ and the morphism σ_Y . Applying ω on the composition of these maps we conclude $\omega(f) = \alpha$, as $\omega(\sigma_Y)$ is injective. Thus ω is full.

It remains to show that ϕ is essentially surjective. For any L-comodule (E,ρ_E) let $E^o\in C$ be such that the sequence

$$0 \longrightarrow E^{o} \longrightarrow E \otimes F \xrightarrow{\delta} E \otimes L \otimes F$$

is exact, where $\delta = \rho_E \otimes id - id \otimes \sigma_F$. Applying ω to this sequence and comparing with (23) we conclude that $\omega(E^o) = E$.

Thus $\omega : \mathfrak{C} \longrightarrow \mathsf{Comod}_f(L)$ is an equivalence of categories. Thus the forgetful functor $\mathsf{Comod}_f(L) \longrightarrow \mathsf{Mod}(R)$ is exact, hence L is flat over R. \Box

Remarks A.1.10. (i) Under the equivalence of Theorem 1.2.2, L, with the right coaction of itself given by the coproduct, corresponds to F. Indeed, this follows from the natural isomorphism

$$\mathsf{Hom}^{\mathsf{L}}(\mathsf{E},\mathsf{L})\simeq\mathsf{Hom}_{\mathsf{R}}(\mathsf{E},\mathsf{R}),\quad\mathsf{f}\mapsto\varepsilon\circ\mathsf{f}.$$

(ii) There is another way to determine L from the category of its comodules as follows. We claim that there is a natural isomorphism

(24)
$$\operatorname{Nat}(\omega, \omega \otimes M) \simeq \operatorname{Hom}_{\mathbb{R}}(L, M),$$

for any R-module M. Indeed, we have

$$\operatorname{Hom}_{\mathbb{R}}(\mathbb{L}, \mathbb{M}) \simeq \operatorname{Hom}(\mathbb{F}, \mathbb{F} \otimes \mathbb{M}) \simeq \operatorname{Hom}(\operatorname{Hom}(\omega, \mathbb{R}), \operatorname{Hom}(\omega, \mathbb{R}) \otimes \mathbb{M}).$$

By means of (14), it suffices to show the isomorphism

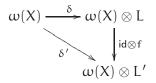
$$\operatorname{Nat}(\omega(X), \omega(X) \otimes M) \simeq \operatorname{Hom}(\operatorname{Hom}(\omega(X), R), \operatorname{Hom}(\omega(X), R) \otimes M)$$

for any $X \in \mathbb{C}^{\circ}$. Since for such X, $\omega(X)$ is finitely generated projective over R, the last isomorphism is obvious. L is usually referred to as the Coend of ω , denoted Coend(ω).

(iii) If $\mathcal{C} = \text{Comod}_f(L)$ and ω is the forgetful functor from \mathcal{C} to Mod(R), then the isomorphism (24) implies that $\text{Coend}(\omega) \simeq L$. Thus a flat coalgebra over R can be *reconstructed* from the category of its comodules.

Remarks A.1.11. Let (\mathcal{C}, ω) and (\mathcal{C}', ω') be two categories satisfying the condition of Theorem 1.2.2 and let $\eta : \mathcal{C} \longrightarrow \mathcal{C}'$ be an R-linear functor such that $\omega'\eta = \omega$. Then η induces a coalgebra homomorphism $f : L \longrightarrow L'$. This can be seen from (24) as follows. The coaction of L' on $\omega'(X')$ defines a natural transformation $\delta' : \omega' \longrightarrow \omega' \otimes L'$. Combine this with η we obtain a natural transformation $\delta : \omega \longrightarrow \omega \otimes L'$. Thus (24) yields a linear map $L \longrightarrow L'$, which satisfies

the following commutative diagram:



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