WEIL q-NUMBERS OF DEGREE 4 OVER Q

Let $\pi \in \mathbf{C}$ be a Weil q-number of degree 4 over $\mathbf{Q}$. We argue that:

(1) For no embedding $\sigma : \mathbf{Q}(\pi) \to \mathbf{C}$ is the image $\sigma(\pi)$ real. This is true because we saw in the talk that in this case π has degree 1 or 2 over $\mathbf{Q}$.

(2) Let $P(X) = X^4 + aX^3 + bX^2 + cX + d$ be the minimal polynomial over $\mathbf{Q}$. Note that we can write

$$P(X) = X^4 + aX^3 + bX^2 + cX + d = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha_4)$$

where α_1, α_2, α_3 and α_4 are all 4 conjugates of π in $\mathbf{C}$.

(3) In particular, for each i we have $\overline{\alpha_i} \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (where $z \mapsto \overline{z}$ is complex conjugation). Since none of the α_i are real we may choose the numbering such that $\overline{\alpha_1} = \alpha_2$ and $\overline{\alpha_3} = \alpha_4$.

(4) Note that $a, b, c, d \in \mathbf{Z}$ because π is an algebraic integer.

(5) Because $|\alpha_i| = \sqrt{q}$ we see that $\overline{\alpha_i} = q/\alpha_i$. In particular we see that the roots of the polynomial $X^4 P(q/X)$ are the same as the roots of $P(X)$. By looking at the leading coefficient we deduce that

$$X^4 P(q/X) = q^2 P(X).$$

Writing this out we obtain

$$q^4 + aq^3X + bq^2X^2 + cqX^3 + dX^4 = q^2X^4 + aq^2X^3 + bq^2X^2 + cq^2X + dq^2.$$

We conclude that $d = q^2$ and that $c = aq$. Thus we conclude that

$$P(X) = X^4 + aX^3 + bX^2 + aqX + q^2.$$

(6) Note that $a = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$. In particular we have

$$|a| \leq 4\sqrt{q}$$

(7) Note that $b = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$. In particular we have

$$|b| \leq 6q$$

(8) Let $\beta = \pi + \overline{\pi} = \pi + q/\pi$. What is the minimal polynomial of β over $\mathbf{Q}$? To see this note that π is a root of $P(X)/X^2$. We write this as

$$\begin{aligned} P(X)/X^2 &= X^2 + aX + b + aq/X + q^2/X^2 \\ &= (X + q/X)^2 - 2q + a(X + q/X) + b \\ &= (X + q/X)^2 + a(X + q/X) + b - 2q \\ &= Y^2 + aY + (b - 2q). \end{aligned}$$

where $Y = X + q/X$. In other words, β satisfies the equation $Y^2 + aY + (b - 2q) = 0$.

(9) In order for π to be a Weil q-number we know by general theory that β has to be totally real. In other words the discriminant Δ of the quadratic polynomial $Y^2 + aY + (b - 2q)$ has to be positive. We compute $\Delta = a^2 - 4(b - 2q) = a^2 + 8q - 4b$. The resulting inequality is $a^2 + 8q > 4b$, or

$$b < a^2/4 + 2q.$$

Note that, in case $b > 0$, this is a stronger inequality than our previous inequality for the magnitude of b.

(10) Note that π is a solution to the equation $X^2 - \beta X + q = 0$. Thus, in order for $\mathbf{Q}(\pi)$ to be a CM field of degree 4 over $\mathbf{Q}$, we need $\beta^2 - 4q$ under any embedding of $\mathbf{Q}(\beta)$ into $\mathbf{R}$ to be negative. In other words we need

$$\left(\frac{-a \pm \sqrt{\Delta}}{2}\right)^2 - 4q < 0$$

(11) We work out what this means:

$$\begin{aligned}
((-a \pm \sqrt{\Delta})/2)^2 - 4q < 0 \quad &\Leftrightarrow \quad ((-a \pm \sqrt{\Delta})/2)^2 < 4q \\
&\Leftrightarrow \quad (-a \pm \sqrt{\Delta})/2 < 2\sqrt{q} \text{ and } (-a \pm \sqrt{\Delta})/2 > -2\sqrt{q} \\
&\Leftrightarrow \quad (-a \pm \sqrt{\Delta}) < 4\sqrt{q} \text{ and } (-a \pm \sqrt{\Delta}) > -4\sqrt{q} \\
&\Leftrightarrow \quad \pm\sqrt{\Delta} < a + 4\sqrt{q} \text{ and } \pm\sqrt{\Delta} > a - 4\sqrt{q} \\
&\Leftrightarrow \quad \sqrt{\Delta} < a + 4\sqrt{q} \text{ and } -\sqrt{\Delta} > a - 4\sqrt{q} \\
&\Leftrightarrow \quad \Delta < a^2 + 8a\sqrt{q} + 16q \text{ and } \Delta < a^2 - 8a\sqrt{q} + 16q \\
&\Leftrightarrow \quad a^2 - 4b + 8q < a^2 - 8|a|\sqrt{q} + 16q \\
&\Leftrightarrow \quad -4b < -8|a|\sqrt{q} + 8q \\
&\Leftrightarrow \quad -b < -2|a|\sqrt{q} + 2q \\
&\Leftrightarrow \quad b > 2|a|\sqrt{q} - 2q
\end{aligned}$$

Note that one of the conclusions of this sequence of inequalities is also that $|a| \leq 4\sqrt{q}$ which we saw before.

(12) So a comlete set of inequalities is the following

$$\begin{array}{rcl}
|a| & \leq & 4\sqrt{q} \\
b & < & a^2/4 + 2q \\
b & > & 2|a|\sqrt{q} - 2q
\end{array}$$