WHY DEGREE 5 MORPHISMS ARE NOT FREE

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We will work throughout over $k = \overline{\mathbb{F}_2}$. Let $R = k[S, T]$. Any homogeneous polynomial of degree 5 in R is of the form $aS^5 + bS^4T + cS^3T^2 + dS^2T^3 + eST^4 + fT^5$. Raising this polynomial to the 5^{th} power we get

The coefficient of the $S^{23}T^2$ term $= a^4c$
The coefficient of the $S^{19}T^6$ term $= b^4c$
The coefficient of the $S^{15}T^{10}$ term $= c^5$
The coefficient of the $S^{11}T^{14}$ term $= d^4c$
The coefficient of the S^7T^{18} term $= e^4c$
The coefficient of the S^3T^{22} term $= f^4c$.

Now suppose we have a morphism $\varphi = (G_0, ..., G_5)$ that is free, where
$G_0 = a_1S^5 + b_1S^4T + c_1S^3T^2 + d_1S^2T^3 + e_1ST^4 + f_1T^5$
$G_1 = a_2S^5 + b_2S^4T + c_2S^3T^2 + d_2S^2T^3 + e_2ST^4 + f_2T^5$
$G_2 = a_3S^5 + b_3S^4T + c_3S^3T^2 + d_3S^2T^3 + e_3ST^4 + f_3T^5$
$G_3 = a_4S^5 + b_4S^4T + c_4S^3T^2 + d_4S^2T^3 + e_4ST^4 + f_4T^5$
$G_4 = a_5S^5 + b_5S^4T + c_5S^3T^2 + d_5S^2T^3 + e_5ST^4 + f_5T^5$
$G_5 = a_6S^5 + b_6S^4T + c_6S^3T^2 + d_6S^2T^3 + e_6ST^4 + f_6T^5$

Then we know that the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{bmatrix}$$

is invertible.

I claim then that the matrix

$$\begin{bmatrix} a_1^4 & b_1^4 & c_1^4 & d_1^4 & e_1^4 & f_1^4 \\ a_2^4 & b_2^4 & c_2^4 & d_2^4 & e_2^4 & f_2^4 \\ a_3^4 & b_3^4 & c_3^4 & d_3^4 & e_3^4 & f_3^4 \\ a_4^4 & b_4^4 & c_4^4 & d_4^4 & e_4^4 & f_4^4 \\ a_5^4 & b_5^4 & c_5^4 & d_5^4 & e_5^4 & f_5^4 \\ a_6^4 & b_6^4 & c_6^4 & d_6^4 & e_6^4 & f_6^4 \end{bmatrix}$$

is also invertible.

It suffices to show that the rows of this matrix are linearly independent, considered as elements of k^6. So, suppose we have $l_1(a_1^4, b_1^4, c_1^4, d_1^4, e_1^4, f_1^4) + ... + l_6(a_6^4, b_6^4, c_6^4, d_6^4, e_6^4, f_6^4)$, where $l_1, ..., l_6 \in k$. Since k is algebraically closed, there exists $m_1, ..., m_6 \in k$ such that $m_i^4 = l_i$. Then, $m_1^4(a_1^4, b_1^4, c_1^4, d_1^4, e_1^4, f_1^4) + ... + m_6^4(a_6^4, b_6^4, c_6^4, d_6^4, e_6^4, f_6^4) = 0$. So, we have

$\sum_i m_i^4 a_i^4 = (\sum_i m_i a_i)^4 = 0 \Rightarrow \sum_i m_i a_i = 0$
$\sum_i m_i^4 b_i^4 = (\sum_i m_i b_i)^4 = 0 \Rightarrow \sum_i m_i b_i = 0$
$\sum_i m_i^4 c_i^4 = (\sum_i m_i c_i)^4 = 0 \Rightarrow \sum_i m_i c_i = 0$
$\sum_i m_i^4 d_i^4 = (\sum_i m_i d_i)^4 = 0 \Rightarrow \sum_i m_i d_i = 0$
$\sum_i m_i^4 e_i^4 = (\sum_i m_i e_i)^4 = 0 \Rightarrow \sum_i m_i e_i = 0$
$\sum_i m_i^4 f_i^4 = (\sum_i m_i f_i)^4 = 0 \Rightarrow \sum_i m_i f_i = 0$

Since the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{bmatrix}$$

is invertible, we must have $m_1 = ... = m_6 = 0$. Hence, $l_1 = ... = l_6 = 0$, and we are done.

Now, since $G_0^5 + ...G_5^5 = 0$, we must have
$a_1^4 c_1 + a_2^4 c_2 + ... + a_6^4 c_6 = 0$ (coeff. of the $S^{23}T^2$ term)
$b_1^4 c_1 + b_2^4 c_2 + ... + b_6^4 c_6 = 0$ (coeff. of the $S^{19}T^6$ term)
$c_1^4 c_1 + c_2^4 c_2 + ... + c_6^4 c_6 = 0$ (coeff. of the $S^{15}T^{10}$ term)
$d_1^4 c_1 + d_2^4 c_2 + ... + d_6^4 c_6 = 0$ (coeff. of the $S^{11}T^{14}$ term)
$e_1^4 c_1 + e_2^4 c_2 + ... + e_6^4 c_6 = 0$ (coeff. of the S^7T^{18} term)
$f_1^4 c_1 + f_2^4 c_2 + ... + f_6^4 c_6 = 0$ (coeff. of the S^3T^{22} term)

We know that the dot product $<,>: k^6 \times k^6 \to k$ is nondegenerate. Since,

$$\begin{bmatrix} a_1^4 & b_1^4 & c_1^4 & d_1^4 & e_1^4 & f_1^4 \\ a_2^4 & b_2^4 & c_2^4 & d_2^4 & e_2^4 & f_2^4 \\ a_3^4 & b_3^4 & c_3^4 & d_3^4 & e_3^4 & f_3^4 \\ a_4^4 & b_4^4 & c_4^4 & d_4^4 & e_4^4 & f_4^4 \\ a_5^4 & b_5^4 & c_5^4 & d_5^4 & e_5^4 & f_5^4 \\ a_6^4 & b_6^4 & c_6^4 & d_6^4 & e_6^4 & f_6^4 \end{bmatrix}$$

is invertible, its columns must be linearly independent, considered as elements of k^6. Hence, $\{(a_1^4, ..., a_6^4), (b_1^4, ..., b_6^4), (c_1^4, ..., c_6^4), (d_1^4, ..., d_6^4), (e_1^4, ..., e_6^4), (f_1^4, ..., f_6^4)\}$ is a basis for k^6. But, then we see that $< (c_1, ..., c_6), (a_1^4, ..., a_6^4) > = < (c_1, ..., c_6), (b_1^4, ..., b_6^4) > = < (c_1, ..., c_6), (c_1^4, ..., c_6^4) > = < (c_1, ..., c_6), (d_1^4, ..., d_6^4) > = < (c_1, ..., c_6), (e_1^4, ..., e_6^4) > = < (c_1, ..., c_6), (f_1^4, ..., f_6^4) > = 0$.

So by the nondegeneracy of $<,>$ it follows that $(c_1, ..., c_6) = (0, ..., 0)$. But this is impossible since

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{bmatrix}$$

is invertible. The contradiction proves no free degree 5 morphism can exist.