7. There are p of the form $(k,1)$ and also $(0,1)$ for a total of $p+1$.

8. Let the three distinct points be $P=(p_0:p_1), Q=(q_0:q_1), R=(r_0:r_1)$ and set the matrix (a_{ij}) be a 2-by-2 matrix. Explicit calculations by multiplying the matrix (a_{ij}) on the three points yields the distinct equations:

$$\begin{aligned} a_{00}p_0 + a_{01}p_1 &= 1 \\ a_{10}p_0 + a_{11}p_1 &= 0 \\ a_{10}q_0 + a_{11}q_1 &= 0 \\ a_{00}r_0 + a_{01}r_1 &= a_{10}r_0 + a_{11}r_1 \end{aligned}$$

yielding the matrix equation

$$\begin{pmatrix} p_0 & p_1 & 0 & 0 \\ 0 & 0 & p_0 & p_1 \\ 0 & 0 & q_0 & q_1 \\ r_0 & r_1 & -r_0 & -r_1 \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the points are pairwise distinct, their homogeneous coordinates cannot be scalar multiples of each other. Therefore, the rows of the matrix are linearly independent and so the matrix is invertible and there exists a solution. Hence, there exists a matrix that transforms three distinct points to $(1:0),(0:1),(1:1)$.

14. Let our conic be $ax^2+by^2+cz^2+dxy+exz+fyz=0$. We dehomogenize by setting $z=1$ to reduce the conic to $ax^2+by^2+c+dxy+ex+fy=0$ and consider the case where p is an odd prime first. If $b=0$, set $x=0$ and then one can easily solve for y. Otherwise, complete the square first with respect to x^2 (by multiplying the entire equation first by a) and making the appropriate substitution to kill the dxy term, ie. $t=ax+\frac{dy}{2}$. Completing the square with respect to the y term and making the appropriate substiution reduces the entire equation to $f(t)=y^2$. If $f(t)$ is of degree 0 or 1, then one can choose an appropriate y to get a solution for t. If f is of degree two, observe that it takes on $\frac{p+1}{2}$ values as t ranges over $\mathbb{F}_p$ (to see this, complete the square). Moreover, there are $p-\frac{p+1}{2}=\frac{p-1}{2}$ non-squares modulo p, so for some t, $f(t)$ is a square, so there is a solution. In either case, there is a solution when p is odd.

If $p=2$, then set $z=1$ again and dehomogenize as before. Suppose that no solutions exist for all pairs (x,y). Then setting $x=0$ forces $b=c=f=1$ and setting $y=0$ forces $a=e=1$. Setting $x=1$ then forces $b=0$. The only conic of this form (ie, that does not have a solution when $z=1$) is $x^2+y^2+xz+yz+z^2$ but this has the solution when $z=0$ of $(1:1:0)$ for example.

So in all cases, there exists a solution.

(This is not a very good writeup at the moment...will clean up later. also, very general result is called Chevalley-warning theorem)

18. An example over $\mathbb{F}_2$: the solution set of $x^3+y^3+z^3=0$ is given by $(1:1:0),(1:0:1),(0:1:1)$ and these points also lie on the curve defined by the equation $x^2y+y^2x+x^2z+z^2x+y^2z+z^2y$, even though the former polynomial does not divide the latter.

Given $n+2$ distinct points in general position in $\mathbb{P}^n$, ie. no $n+1$ subset of points lie on a hyperplane, there exists a change of coordinates sending them to $[1:\cdots:0],[0:\cdots:1],[1:\cdots:1]$.

Proof: Call the $n+2$ points $x_0=[x_0^0:\cdots:x_0^n],\ldots,x_{n+1}=[x_{n+1}^0:\cdots:x_{n+1}^n]$. Then by the transitivity of the $PGL(k)$, there exists a transformation sending $[1:\cdots:0],\ldots,[0:\cdots:1]$ to the points $x_0,\ldots,x_n$. We can write this explicitly as the $(n+1)\times(n+1)$ matrix whose columns are given by the entries of the $x_0,\ldots,x_n$. Then, it suffices to show that a matrix whose columns are scalar multiples of the columns of this matrix send the point $(1:\cdots:1)$ to $x_{n+1}=[x_{n+1}^0,\cdots,x_{n+1}^n]$. Writing this out shows that we only need a solution to the linear system of equations in $n+1$ variables (the scaling factors) in $n+1$ equations (whose coefficients arise as the coordinates of the $x_0,\ldots,x_n$). But since the $x_0,\ldots,x_n$ are linearly independent in the vector space k^{n+1}, a solution always exists and so our transformation also always exists.