Notes on Representations of Finite Groups, Fall 2025

List of topics we will discuss in the lectures. This document will be updated during the semester, so please make sure to reload the page before downloading. The content of the course is what is discussed during lectures, so you are encouraged to take notes.

1. Linear algebra

(1) Vector spaces and linear maps. There is a general notion of a vector space V over a field F, but we will mostly work with vector spaces over the real numbers, so $F = \mathbf{R}$, or over the complex numbers, so $F = \mathbf{C}$.

(2) A linear map $A : V \to V$ is called *diagonalizable* if there exists a basis of V consisting of eigenvectors of A. An $n \times n$ square matrix A is called diagonalizable if there exists an invertible matrix P such that PAP^{-1} is diagonal; this is equivalent to asking the linear map $F^n \to F^n$ defined by A to be diagonalizable.

(3) A vector space V over a field F has a dimension. Most of our vector spaces have finite dimension.

(4) Let V be a finite dimensional vector space over F, say of dimension d. Let $A : V \to V$ be a linear map. Then we have the *characteristic polynomial* $p_A(t)$ of A. This is a monic polynomial of degree d with coefficients in F. The roots of $p_A(t)$ are the eigenvalues of A. It is defined as

$$p_A(t) = \det\left(t\mathrm{id}_V - A\right)$$

(5) In the situation above, the *trace* of A is the negative of the coefficient of t^{d-1} in $p_A(t)$ and the *determinant* of A is $(-1)^d$ times the constant term of $p_A(t)$.

(6) Jordan normal form and relation to $p_A(t)$, especially for $F = \mathbf{R}$ or $F = \mathbf{C}$. For example, suppose that $A : V \to V$ is a linear map and that there exists a basis $v_1, \ldots, v_n$ of V consisting of eigenvectors for A, i.e., $A(v_i) = \alpha_i v_i$. Then $p_A(t) = \prod(t - \alpha_i)$.

(7) Given two vector spaces V and W over F we can make some other vector spaces, for example
   (a) the direct sum $V \oplus W$,
   (b) the tensor product $V \otimes W$,
   (c) the space $Hom(V, W)$ of F-linear maps from V to W, and
   (d) a special case of (c) is the dual $V^* = Hom(V, F)$ of V, i.e., the space of linear functionals $\lambda : V \to F$.

(8) In (6) assume we are also given linear maps $A : V \to V$ and $B : W \to W$. We obtain corresponding linear maps $A \oplus B : V \oplus W \to V \oplus W$, $A \otimes B : V \otimes W \to V \otimes W$, "$Hom(A, B)$": $Hom(V, W) \to Hom(V, W)$, and $A^* : V^* \to V^*$.

(9) In (7) assume V and W are finite dimensional and $F = \mathbf{C}$. Write $p_A(t) = \prod_i (t - \alpha_i)$ and $p_B(t) = \prod_j (t - \beta_j)$. We can express the chacteristic polynomials, the traces, and the determinants of the maps $A \oplus B$, $A \otimes B$, and "$Hom(A, B)$" in terms of $p_A(t)$ and $p_B(t)$ as follows:
   (a) $p_{A \oplus B}(t) = p_A(t)p_B(t)$, $\mathrm{Trace}(A \oplus B) = \mathrm{Trace}(A) + \mathrm{Trace}(B)$, and $\det(A \oplus B) = \det(A) + \det(B)$

(b) $p_{A\otimes B}(t) = \prod_{i,j}(t - \alpha_i\beta_j)$, $\mathrm{Trace}(A \otimes B) = \mathrm{Trace}(A)\mathrm{Trace}(B)$, and $\det(A \oplus B) = \det(A)^{\dim(W)} \det(B)^{\dim(V)}$
(c) $p_{Hom(A,B)}(t) = \prod_{i,j}(t-\alpha_i\beta_j)$, $\mathrm{Trace}(Hom(A, B)) = \mathrm{Trace}(A)\mathrm{Trace}(B)$, and $\det(Hom(A, B)) = \det(A)^{\dim(W)} \det(B)^{\dim(V)}$
(d) $p_{A^*}(t) = p_A(t)$, $\mathrm{Trace}(A^*) = \mathrm{Trace}(A)$, and $\det(A^*) = \det(A)$.

(10) An *inner product* on a real vector space V is a function $\langle -, - \rangle : V \times V \to \mathbf{R}$ which is linear in both variables, symmetric, and satisfies $\langle v, v\rangle > 0$ for $v \in V$ nonzero.
(11) The standard example of an inner product is to take $V = \mathbf{R}^n$ and $\langle v, w\rangle = v_1w_1 + \ldots + v_nw_n$.
(12) A *Hermitian inner product* on a complex vector space V is a function $\langle -, -\rangle : V \times V \to \mathbf{R}$ which is linear in the first variable, semilinear in the second variable[1], is Hermitian, i.e., $\langle w, v\rangle = \overline{\langle v, w\rangle}$ for all $v, w \in V$, and satisfies $\langle v, v\rangle > 0$ for $v \in V$ nonzero.
(13) The standard example of an Hermitian inner product is to take $V = \mathbf{C}^n$ and $\langle v, w\rangle = v_1\overline{w_1} + \ldots + v_n\overline{w_n}$.
(14) The Gram-Schmidt algorithm works for both inner products and Hermitian inner products on finite dimensional vector spaces V. It follows from this that, if $W \subset V$ is a subspace, then we always have a direct sum decomposition

$$V = W \oplus W^\perp$$

where $W^\perp = \{v \in V \mid \langle w, v\rangle = 0\ \forall w \in W\}$ is the orthogonal complement of W in V.

2. Orthogonal groups, Unitary groups, Quaternions

Notation: given an integer $n \geq 1$ we denote

(1) $O(n)$ the group of $n \times n$ orthogonal matrices,
(2) $SO(n)$ the subgroup of $O(n)$ consisting of orthogonal matrices whose determinant is 1,
(3) $U(n)$ the group of $n \times n$ unitary matrices,
(4) $SU(n)$ the subgroup of $U(n)$ consisting of unitary matrices whose determinant is 1.

Finite subgroups of $SO(3)$: a finite subgroup of $SO(3)$ is isomorphic either to a cyclic group, a dihedral group, or the rotational symmetry group of one of the regular solids

The quaternions H. This is a (noncommutative) **R**-algebra which as an **R**-vector space has the basis $1, i, j, k$. In other words, we have

$$\mathbf{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbf{R}\}$$

[1]This means that $\langle v, \lambda w\rangle = \bar{\lambda}\langle v, w\rangle$ for $\lambda \in \mathbf{C}$ and $v, w \in V$.

The associative multiplication is characterized by the rules $ij = k$ and $ji = -k$ and $i^2 = j^2 = k^2 = -1$. Written out this gives

$$\begin{aligned}(a + bi + cj + dk)(a' + b'i + c'j + d'k) = &(aa' - bb' - cc' - dd')+\\ &(ab' + ba' + cd' - dc')i+\\ &(ac' + ca' - bd' + db')j+\\ &(ad' + da' + bc' - cb')k\end{aligned}$$

The conjugate of a quaternion $q = a + bi + cj + dk$ is

$$\overline{q} = a - bi - cj - dk$$

Conjugation on the quaternions is an involution in the sense that it is additive (in fact $\mathbf{R}$-linear) and satisfies $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$.

Det and Trace. From the above definition one sees that

$$\mathrm{Norm}(q) = q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2$$

is a nonnegative real scalar positive if $q \neq 0$ similar to what happens for complex numbers. We call $q\overline{q}$ the *norm*[2] or *determinant* of the quaternion and we call

$$\mathrm{Trace}(q) = q + \overline{q} = 2a$$

the *trace* of the quaternion q. The Norm is multiplicative and the Trace is additive, i.e., we have $\mathrm{Norm}(qp) = \mathrm{Norm}(q)\mathrm{Norm}(p)$ and we have $\mathrm{Trace}(q + p) = \mathrm{Trace}(q) + \mathrm{Trace}(p)$. The trace also satisfies $\mathrm{Trace}(qp) = \mathrm{Trace}(pq)$.

Groups of invertible elements. Since $q\overline{q} = \mathrm{Norm}(q)$ is a positive scalar if q is nonzero, we see that every nonzero quaternion is invertible with inverse

$$q^{-1} = \mathrm{Norm}(q)^{-1}\overline{q}$$

In other words the (noncommutative) group of units $\mathbf{H}^*$ of $\mathbf{H}$ is equal to $\mathbf{H} \setminus \{0\}$. Unit quaternions. By the above, we see that the set of quaternions q of norm 1 form a subgroup of $\mathbf{H}^*$:

$$\{q \in \mathbf{H} : \mathrm{Norm}(q) = 1\} \subset \mathbf{H}^*$$

Inner conjugation. For a nonzero quaternion q consider the map "conjugation with q"

$$\mathrm{inn}_q : \mathbf{H} \to \mathbf{H}, \quad p \mapsto qpq^{-1}$$

Note that

$$\mathrm{inn}_{qq'} = \mathrm{inn}_q \circ \mathrm{inn}_{q'}$$

hence we get a homomorphism from $\mathbf{H}^*$ into the group of $\mathbf{R}$-linear automorphisms of $\mathbf{H}$ itself. The map inn_q preserves the algebra structure and preserves the trace and the norm because

$$\mathrm{Trace}(qpq^{-1}) = \mathrm{Trace}(q^{-1}qp) = \mathrm{Trace}(p)$$

and similarly for Norm. Thus inn_q preserves lengths hence defines an element of $O(4)$ and a calculation shows it is even in $SO(4)$. However, inn_q also preserves the subspace of elements whose trace is 0 and acts as the identity on its orthogonal complement (the multiples of 1 in $\mathbf{H}$). Thus we obtain a map

$$\mathbf{H}^* \longrightarrow SO(3), \quad q \mapsto \mathrm{inn}_q \text{ restricted to } \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$$

[2] In some references the square root of our norm is used.

Fact. The displayed map above determines a map

$$\text{unit quaternions} = \{q \in \mathbf{H} : \text{Norm}(q) = 1\} \longrightarrow SO(3)$$

which is surjective with kernel the group $\{\pm 1\}$.

Quaternions as complex matrices. There is an algebra homomorphism

$$\mathbf{H} \longrightarrow \text{Mat}(2 \times 2, \mathbf{C}), \quad a + bi + cj + dk \longmapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

Under this map the Norm and Trace of a quaternion match the determinant and trace of the corresponding matrix (and the conjugate of a quaternion maps to the conjugate transpose of the matrix). In particular, this determines a homomorphism

$$\mathbf{H}^* \longrightarrow GL_2(\mathbf{C})$$

Via this homomorphism we get an isomorphism

$$\text{unit quaternions} = \{q \in \mathbf{H} : \text{Norm}(q) = 1\} \cong SU(2)$$

Upshot. There is a surjective group homomorphism $SU(2) \to SO(3)$ whose kernel is $\{\pm 1\}$.

Finite subgroups of $SU(2)$. A finite subgroup of $SU(2)$ is either: if of odd order, then maps isomorphically to an odd order finite subgroup of $SO(3)$, or if even order, then it is a double cover (full inverse image) of a finite subgroup of $SO(3)$ via the map above.

Finite subgroups of $SL_2(\mathbf{C})$ or $GL_2(\mathbf{C})$. The finite subgroups of $SL_2(\mathbf{C})$ (up to conjugacy) are the same thing as the finite subgroups of $SU(2)$ (up to conjugacy). Similar for $GL_2(C)$ and $U(2)$.

Finite subgroups of $U(2)$. There is a map $U(2) \to SO(3)$ because $U(2)$ modulo it's center is the same as $SU(2)/\{\pm 1\} = SO(3)$. Thus if G is a finite subgroup of $U(2)$, then its image in $SO(3)$ is on our list above. Then one can work out the possibilities for G from that[3]. See for example Falbel, E., Paupert, J. *Fundamental Domains for Finite Subgroups in U(2) and Configurations of Lagrangians*, Geom Dedicata 109, 221–238 (2004). `https://doi.org/10.1007/s10711-004-2455-2`.

3. Some group theory

(1) Elementary concepts: groups, homomorphisms of groups, subgroups, normal subgroups, products, semi-direct products, group actions, orbits, centralizer of an element, center of a group, conjugacy in a group (conjugate elements, conjugacy classes), abelian (or commutative) groups, commutator subgroup and the abelianization of a group, and simple groups.
(2) Most of our groups will be finite groups.
(3) We will allude to p-groups and p-Sylow subgroups (definition, existence, conjugacy) in finite groups.
(4) Examples of groups:
    (a) cyclic groups: either infinite cyclic $\mathbf{Z}$, sometimes denoted C_∞, or finite cyclic of order n, i.e., $\mathbf{Z}/n\mathbf{Z}$ (integers modulo n) or μ_n (nth roots of unity), sometimes the notation C_n is used.

[3]This description in particular tells us that G modulo its center is a subgroup of $SO(3)$. Hence, for example. only a finite number of noncommutative simple groups arise as subgroups of $U(2)$.

(b) finitely generated abelian groups and structure theorem,
(c) finite abelian groups and there shapes,
(d) dihedral groups D_n of order $2n$; these are the symmetries of a regular n-gon and generated by two elements ρ and τ where ρ is a rotation over $2\pi/n$ and τ is a reflection in the x-axis,
(e) groups of low orders 1,2,3,4,5,6, etc
(f) symmetric groups S_n, permutation matrices, generated by transpositions, products of disjoint cycles (unique up to reordering), conjugacy classes, sign of a permutation
(g) alternating groups A_n, generated by 3-cycles, simple for $n \geq 5$
(h) automorphism groups of mathematical objects
(i) subgroups of $GL_n(\mathbf{R})$ and $GL_n(\mathbf{C})$, for example the "Heisenberg group",
(j) Inside the unit quaternions we have the finite groups
    (i) quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ of order 8
    (ii) binary tetrahedral group $2T = \langle 2, 3, 3\rangle$ defined as $Q_8 \rtimes C_3$ where C_3 is the order 3 cyclic group generated by the quaternion $-\frac{1}{2}(1+i+j+k)$. It turns out that the elements of $2T$ which are not in Q_8 are the elements $\frac{1}{2}(\pm 1 + \pm i + \pm j + \pm k)$. Thus $2T$ has order $8 + 16 = 24$
    (iii) binary octahedral group $2O = \langle 2, 3, 4\rangle$ defined as

$$2O = 2T \bigcup \left\{ \frac{1}{\sqrt{2}}(\pm 1 + \pm i), \frac{1}{\sqrt{2}}(\pm 1 + \pm j), \frac{1}{\sqrt{2}}(\pm 1 + \pm k), \frac{1}{\sqrt{2}}(\pm i + \pm j), \frac{1}{\sqrt{2}}(\pm i + \pm k), \frac{1}{\sqrt{2}}(\pm j + \pm k) \right\}$$

    of order 48.
(k) finite groups of Lie type, eg $SL_2(\mathbf{F}_q)$

4. Representations

(1) definition of a representation, maps of representations, category of representations,
(2) new representations from old ones: direct sum, tensor product, dual representation, and Hom representation,
(3) the group algebra of a group G over $\mathbf{Z}$, over $\mathbf{R}$, over $\mathbf{C}$. We will denote this $\mathbf{Z}[G]$, or $\mathbf{R}[G]$, or $\mathbf{C}[G]$,
(4) relation between representations and modules over the group algebra,
(5) examples
    (a) trivial representation,
    (b) 1-dimensional representations and G^{ab},
    (c) standard representation of S_n,
    (d) permutation representations,
    (e) 2-dimensional representation of S_4 using $S_4 \to S_3$.
    (f) Quaternions as complex 2×2-matrices and the complex representations of Q_8, $2T$ and $2O$ in dimension 2,
    (g) regular representation of G,
    (h) representations of the infinite cyclic group and JNF

(6) invariant subspaces, decomposable representations, irreducible representations, completely reducible representations,
(7) the invariants V^G in a representation V of G,
(8) canonical projection onto V^G,
(9) Maschke's theorem; two proofs
(10) representations of finite abelian groups,
(11) Schur's lemma over **C**; counterexample over **R**.

5. The character of a representation

(1) the character of a representation,
(2) a character is a class function,
(3) the value of a character at the neutral element,
(4) the value of a character at g^{-1},
(5) character of a 1-dimensional representation,
(6) character of direct sums, duals, hom, and tensor products of represenations,
(7) examples of characters
  (a) character of the trivial representation
  (b) character of a permutation representation
  (c) character of the standard representation of S_n
  (d) characters of the 2-dimensional representations of dihedral groups, of Q_8, $2T$, and $2O$,
(8) the dimension of V^G in terms of the character,
(9) orthogonality relations between characters,
(10) decomposition into irreducibles and characters,
(11) character determines isomorphism class.
(12) Example: decomposing the regular representation.
(13) $|G| = \sum d_i^2$
(14) consequences of permutation representations (other pdf Bob)
(15) irreducibility for small standard representation for S_n for $n \geq 2$ and for A_n for $n \geq 4$.

6. Characters, part 2

(1) dimension of the space of class functions is the number of conjugacy classes
(2) main theorem on the number of irreducible representations and their characters
(3) special formulae for character values
(4) character tables

7. Fourier transform

Skip this section?

8. Characters, part 3

(1) representations of product groups
(2) Theorem of Frobenius
(3) Theorem of Burnside