SHEAVES ON THE SPECTRUM OF A RING

Throughout A is a ring. We set $X = \mathrm{Spec}(A)$. We denote $\mathcal{B}$ the set of principal opens of X (AKA standard opens). In a formula

$$\mathcal{B} = \{U \subset X \mid \exists\ f \in A,\ U = D(f)\}$$

Sheaves on a basis: see Tag 009H

Think of $\mathcal{B}$ as a category: the objects are the elements of $\mathcal{B}$ and the morphisms are the inclusions. A *presheaf* $\mathcal{F}$ *on* $\mathcal{B}$ is a contravariant functor from $\mathcal{B}$ to the category of sets (or abelian groups, rings, etc). Similarly for presheaves of modules over a given presheaf of rings. We say $\mathcal{F}$ is a *sheaf on* $\mathcal{B}$ if and only if for every covering[1]

$$\mathcal{U} : U = U_1 \cup \ldots \cup U_n$$

with $U, U_i \in \mathcal{B}$ we have that

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram. Observe that this makes sense because

$$U, U' \in \mathcal{B} \Rightarrow U \cap U' \in \mathcal{B}$$

as you can easily verify.

Lemma 0.1. *The category of sheaves on X and sheaves on $\mathcal{B}$ are equivalent via the functor which takes a sheaf on X and restricts it to $\mathcal{B}$.*

Proof. This is a copy of Tag 009O □

If $\mathcal{F}|_\mathcal{B}$ denotes the restriction of $\mathcal{F}$ on X to $\mathcal{B}$ (as in the equivalence of Lemma 0.1), then we have for $x \in X$ the equality

$$\mathcal{F}_x = \mathrm{colim}_{x \in U \in \mathcal{B}} \mathcal{F}(U) = \mathrm{colim}_{x \in U \in \mathcal{B}} \mathcal{F}|_\mathcal{B}(U)$$

Hence we can directly compute the stalks in terms of the sheaf on $\mathcal{B}$.

The structure sheaf: see Tag 01HR

For every $U \in \mathcal{B}$ choose an element $f \in A$ such that $U = D(f)$. The we set

$$\mathcal{O}_X(U) = A_f$$

If $U = D(f) \supset V = D(g)$, then we can write $g^n = af$ for some $n > 0$ and $a \in A$ (small detail omitted) and we define the restriction mapping for $\mathcal{O}_X$ as the map of A-algebras

$$\mathcal{O}_X(U) = A_f \longrightarrow A_g = \mathcal{O}_X(V)$$

sending b/f^m to ba^m/g^{nm}. It is easy to see this is a presheaf of rings.

The sheaf of modules $\widetilde{M}$ associate to an A-module M: see Tag 01HR

Let M be an A-module. For every $U \in \mathcal{B}$ choose an element $f \in A$ such that $U = D(f)$. The we set

$$\widetilde{M}(U) = M_f$$

[1]Since every element of $\mathcal{B}$ is quasi-compact we only need to consider finite coverings.

If $U = D(f) \supset V = D(g)$, then we can write $g^n = af$ for some $n > 0$ and $a \in A$ (small detail omitted) and we define the restriction mapping for $\widetilde{M}$ as the map of A-modules

$$\widetilde{M}(U) = M_f \longrightarrow M_g = \widetilde{M}(V)$$

sending x/f^m to $a^m x/g^{nm}$. It is easy to see this is a presheaf of modules over $\mathcal{O}_X$. Also, observe that $\mathcal{O}_X = \widetilde{A}$, and hence if we prove $\widetilde{M}$ is a sheaf, then the same thing holds for $\mathcal{O}_X$.

Sheaf property: To check the sheaf property consider a covering

$$\mathcal{U} : U = U_1 \cup \ldots \cup U_n$$

with $U, U_i \in \mathcal{B}$. Write $U = D(f)$ and $U_i = D(f_i)$. To check the sheaf property it suffices to check

$$0 \to M_f \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots$$

is exact. The fact that $U = \bigcup U_i$ implies that

$$M_{f_{i_0} \ldots f_{i_p}} = (M_f)_{f_{i_0} \ldots f_{i_p}}$$

and that $f_1, \ldots, f_n$ generate the unit ideal in the ring A_f. Hence the alternating Čech complex for $\mathcal{U}$ and $\widetilde{M}$ is the complex of Lemma 0.2 for the ring A_f, the module M_f, and the elements $f_1/1, \ldots, f_n/1$ of A_f.

Lemma 0.2. *Let A be a ring, let M be an A-module, let $f_1, \ldots, f_n \in A$ generate the unit ideal. Then the complex*

$$0 \to M \to \prod\nolimits_{i_0} M_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \to \ldots$$

is exact.

Proof. Two steps: first if f_i is a unit in A for some i, then one writes an explicit homotopy, next, one proves the lemma below to deduce the general case from this special case. □

Lemma 0.3. *Let A be a ring, let $M_1 \to M_2 \to M_3$ be a complex of A-modules, let $f_1, \ldots, f_n \in A$ generate the unit ideal. Then $M_1 \to M_2 \to M_3$ is exact if and only if for each i the complex*

$$(M_1)_{f_i} \to (M_2)_{f_i} \to (M_3)_{f_i}$$

is exact.

Proof. Using that localization is exact this reduces to the statement: if an A-module H satisfies $H_{f_i} = 0$ for $i = 1, \ldots, n$, then $H = 0$. This is proved by considering an element $x \in H$ and observing that the annihilator ideal of x contains $f_i^{N_i}$ for some $N_i > 0$. Since $f_1^{N_1}, \ldots, f_n^{N_n}$ is the unit ideal of A, we conclude that $x = 0$. See Tag 00EN for more results of this nature. □

Proposition 0.4. *The higher cohomology groups of the structure sheaf $\mathcal{O}_X$ and of the sheaves $\widetilde{M}$ vanish.*

Proof. The discussion and lemmas above show that for any open covering $\mathcal{U} : U = U_1 \cup \ldots \cup U_n$ with $U, U_i \in \mathcal{B}$ the higher Čech cohomology of $\widetilde{M}$ vanishes. Thus we may apply Tag 01EW. See also Tag 01XB. □