Schemes

Exercises 6

Schemes – Examples again

Feel free to quote results from Hartshorne or EGA.

1. Čech cohomology. Here k is a field.

(a) Let X be a scheme with an open covering $\mathcal{U} : X = U_1 \cup U_2$, with $U_1 = \operatorname{Spec} k[x]$, $U_2 = \operatorname{Spec} k[y]$ with $U_1 \cap U_2 = \operatorname{Spec} k[z, 1/z]$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We've seen in the lectures that such an X exists; it is the affine line zith zero doubled.) Compute $\check{H}^1(\mathcal{U}, \mathcal{O})$; eg. give a basis for it as a k-vectorspace.

(b) For each element in $\check{H}^1(\mathcal{U}, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$

such that the boundary $\delta(1) \in \check{H}^1(\mathcal{U}, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. It is also OK to show abstractly such a thing has to exist.)

Definition of delta. Suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of abelian sheaves on any topological space X. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to \check{H}^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that for each i there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of τ to U_i. Then consider the assignment

$$(i_0, i_1) \longmapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$

This is clearly a 1-coboundary in the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

2. Algebra. (Silly and should be easy.)

(a) Give an example of a ring A and a nonsplit short exact sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

(b) Give an example of a nonsplit sequence of A-modules as above and a faithfully flat $A \to B$ such that

$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0.$$

is split as a sequence of B-modules.

3. Maps of Proj. Let R and S be graded rings. So $R = \oplus_{d \geq 0} R_d$ and $R_a \cdot R_b \subset R_{a+b}$. Suppose we have a ring map

$$\varphi : R \to S$$

such that there exists an integer $e \geq 1$ such that $\varphi(R_d) \subset S_{de}$.

(a) For which elements $\mathfrak{p} \in \operatorname{Proj}(S)$ is there a well-defined corresponding point in $\operatorname{Proj}(R)$? In other words, find a suitable open $U \subset \operatorname{Proj}(S)$ such that φ defines a continuous map $\operatorname{Proj}(\varphi) : U \to \operatorname{Proj}(R)$.

(b) Give an example where $U \neq \operatorname{Proj}(S)$.

(c) Give an example where $U = \operatorname{Proj}(S)$.

(d) (Do not write this down.) Convince yourself that the continuous map $U \to \operatorname{Proj}(R)$ comes canonically with a map on sheaves so that $\operatorname{Proj}(\varphi)$ is a morphism of schemes:

$$\operatorname{Proj}(S) \supset U \longrightarrow \operatorname{Proj}(R)$$

Notation. Let R be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \mathrm{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on X such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D_+(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (=elements homogenous of degree n in $R_f = R[1/f]$). See Hartshorne, page 116+. Note that there are natural maps

$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

4. Pathologies in Proj. Give examples of R as above such that
- **(a)** $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.
- **(b)** $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is NOT an isomorphism.