Schemes

Exercises 5

Schemes – invertible sheaves are important

Feel free to quote results from Hartshorne or EGA.

An invertible $\mathcal{O}_X$-module on a locally ringed space $(X, \mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that every point has an open neighbourhood $U \subset X$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$ as $\mathcal{O}_U$-module. We say that $\mathcal{L}$ is trivial if it is isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module.

1. General facts.
  (a) Show that an invertible $\mathcal{O}_X$-module on a scheme X is quasi-coherent.
  (b) Suppose $X \to Y$ is a morphism of ringed spaces, and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Show that $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$ module.

2. Algebra.
  (a) Show that an invertible $\mathcal{O}_X$-module on an affine scheme $\operatorname{Spec} A$ corresponds to an A-module M which is (i) finite, (ii) projective, (iii) locally free of rank 1, and hence (iv) flat, and (v) finitely presented. (Feel free to quote things from last semesters course; or from algebra books.)
  (b) Suppose that A is a domain and that M is a module as in (a). Show that M is isomorphic as an A-module to an ideal $I \subset A$ such that $IA_{\mathfrak{p}}$ is principal for every prime $\mathfrak{p}$.

3. Simple examples.
  (a) Let k be a field. Let $A = k[x]$. Show that $X = \operatorname{Spec} A$ has only trivial invertible $\mathcal{O}_X$-modules.
  (b) Let A be the ring

$$A = \{f \in k[x] \mid f(0) = f(1)\}.$$

  Show that $X = \operatorname{Spec} A$ has a nontrivial invertible $\mathcal{O}_X$-module, unless $k = \mathbb{F}_2$. (Hint: Think about $\operatorname{Spec} A$ as identifying 0 and 1 in $\mathbb{A}^1_k = \operatorname{Spec} k[x]$.)
  (c) Same question for the ring $A = k[x^2, x^3] \subset k[x]$ (except now $k = \mathbb{F}_2$ works as well).

4. Higher dimensions.
  (a) Prove that every invertible sheaf on two dimensional affine space is trivial. More precisely, let $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ where k is a field. Show that every invertible sheaf on $\mathbb{A}^2_k$ is trivial. (Hint: One way to do this is to consider the corresponding module M, to look at $M \otimes_{k[x,y]} k(x)[y]$, and then use 3(a) to find a generator for this; then you still have to think. Another way to is to use 2(b) and use what we know about ideals of the polynomial ring: primes of height one are generated by an irreducible polynomial; then you still have to think.)
  (b) Prove that every invertible sheaf on any open subscheme of two dimensional affine space is trivial. More precisely, let $U \subset \mathbb{A}^2_k$ be an open subscheme where k is a field. Show that every invertible sheaf on U is trivial. Hint: Show that every invertible sheaf on U extends to one on $\mathbb{A}^2_k$. Not easy; but you can find it in Hartshorne.
  (c) Find an example of a nontrivial invertible sheaf on a punctured cone over a field. More precisely, let k be a field and let $C = \operatorname{Spec} k[x, y, z]/(xy - z^2)$. Let $U = C \setminus \{(x, y, z)\}$. Find a nontrivial invertible sheaf on U. Hint: It may be easier to compute the group of isomorphism classes of invertible sheaves on U than to just find one. Note that U is covered by the opens $\operatorname{Spec} k[x, y, z, 1/x]/(xy - z^2)$ and $\operatorname{Spec} k[x, y, z, 1/y]/(xy - z^2)$ which are "easy" to deal with.