Schemes

Exercises 2

Schemes – examples are important

Let $\mathcal{C}$ be the category of locally ringed spaces. An affine scheme is an object in $\mathcal{C}$ isomorphic in $\mathcal{C}$ to a pair of the form $(\operatorname{Spec} A, \mathcal{O}_A)$. A scheme is an object $(X, \mathcal{O}_X)$ of $\mathcal{C}$ such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

1. Suppose that X is a scheme whose underlying topological space has 2 points. Show that X is an affine scheme.
2. Give an example of an affine scheme $(X, \mathcal{O}_X)$ and an open $U \subset X$ such that $(U, \mathcal{O}_X|U)$ is not an affine scheme.
3. Given an example of a pair of affine schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, an open subscheme $(U, \mathcal{O}_X|_U)$ of X and a morphism of schemes $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ that does not extend to a morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.
4. Give an example of a scheme X, a field K, and a morphism of ringed spaces $\operatorname{Spec} K \to X$ which is NOT a morphism of schemes.
5. Do all the exercises in Hartshorne, Chapter II, Sections 1 and 2... Just kidding!

Remark. When $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open subset then $(U, \mathcal{O}_X|_U)$ is a ringed space. Notation: $\mathcal{O}_U = \mathcal{O}_X|_U$. There is a canonical morphisms of ringed spaces

$$j : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X).$$

If $(X, \mathcal{O}_X)$ is a locally ringed space, so is $(U, \mathcal{O}_U)$ and j is a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme so is $(U, \mathcal{O}_U)$ and j is a morphism of schemes. We say that $(U, \mathcal{O}_U)$ is an *open subscheme* of $(X, \mathcal{O}_X)$ and that j is an *open immersion.* More generally, any morphism $j' : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ that is *isomorphic* to a morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ as above is called an open immersion.