# LECTURE NOTES B

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## 1. Dualizing sheaf

Let $k$ be a field. Let $X$ be a proper scheme over $k$. We say a pair $(\omega_X, t)$ is a *dualizing sheaf* or *dualizing module* if $\omega_X$ is a coherent $\mathcal{O}_X$-module and

$$t : H^{\dim X}(X, \omega_X) \longrightarrow k$$

is a $k$-linear map such that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ the pairing

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \times H^{\dim X}(X, \mathcal{F}) \longrightarrow k, \quad (\varphi, \xi) \longmapsto t(\varphi(\xi))$$

is a perfect pairing of finite dimensional $k$-vector spaces.

**Theorem 1.1.** *If $X$ is projective over $k$ then there exists a dualizing sheaf. In fact, for any closed immersion $i : X \to P = \mathbf{P}^n_k$ there is an isomorphism*

$$i_*\omega_X = \mathcal{E}xt^{n-\dim X}_{\mathcal{O}_P}(i_*\mathcal{O}_X, \omega_P)$$

**Lemma 1.2.** *If $X \subset \mathbf{P}^n_k$ is a hypersurface of degree $d$ then $\omega_X = \mathcal{O}_X(d - n - 1)$.*

## 2. Projective schemes

Let $X \subset \mathbf{P}^n_k$ be a closed subscheme. Recall that if $i : X \to \mathbf{P}^n_k$ is the corresponding closed immersion, then we use the notation

$$\mathcal{O}_X(m) = i^*\mathcal{O}_{\mathbf{P}^1_k}(m)$$

Recall that $\mathcal{O}_X(1)$ is an ample invertible sheaf on $X$. Also, since $i^*$ commutes with tensor product we have $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$. By the projection formula we have

$$i_* \left(\mathcal{O}_X(m)\right) = (i_*\mathcal{O}_X) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_{\mathbf{P}^1_k}(m)$$

This also means the notation $i_*\mathcal{O}_X(m)$ is not ambiguous.

For every integer $m$ there is a surjection of sheaves

$$\mathcal{O}_{\mathbf{P}^n_k}(m) \longrightarrow i_*\mathcal{O}_X(m)$$

The *homogeneous ideal of* $X$ is the graded ideal $I \subset k[T_0, \ldots, T_n]$ with degree $m$ part equal to the kernel of

$$k[T_0, \ldots, T_n]_m = \Gamma(\mathbf{P}^1_k, \mathcal{O}_{\mathbf{P}^1_k}(m)) \longrightarrow \Gamma(\mathbf{P}^1_k, i_*\mathcal{O}_X(m)) = \Gamma(X, \mathcal{O}_X(m))$$

**Lemma 2.1.** *In the situation above the surjection of graded rings*

$$k[T_0, \ldots, T_n] \longrightarrow k[T_0, \ldots, T_n]/I$$

*induces a closed immersion*

$$\mathit{Proj}(k[T_0, \ldots, T_n]/I) \longrightarrow \mathit{Proj}(k[T_0, \ldots, T_n]) = \mathbf{P}^1_k$$

*whose image is equal to* $X$.

## 3. Ext pairings

Let $(X, \mathcal{O}_X)$ be a ringed space. Given $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$ there is a pairing

$$\mathrm{Ext}^n_X(\mathcal{G}, \mathcal{H}) \times \mathrm{Ext}^m_X(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^{n+m}_X(\mathcal{F}, \mathcal{H}), \quad (\eta, \xi) \longmapsto \eta \circ \xi$$

One way to define these pairings is using injective resolutions. For example choose injective resolutions $\mathcal{G} \to \mathcal{J}^\bullet$ and $\mathcal{H} \to \mathcal{I}^\bullet$. We may represent $\xi$ and $\eta$ by maps of complexes

$$\tilde{\xi} : \mathcal{F} \to \mathcal{J}^\bullet[m] \quad \tilde{\eta} : \mathcal{G} \to \mathcal{I}^\bullet[n]$$

as discussed previously. Then you can find the dotted arrow making the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\tilde{\eta}"'] \arrow[d] & \mathcal{I}^\bullet[n] \\ \mathcal{J}^\bullet \arrow[ur, dotted, "\tilde{\eta}'"'] & \end{tikzcd}$$

commute. See Lemma 013P. Here we use that the vertical arrow is an (injective) quasi-isomorphism of complexes and that the complex $\mathcal{I}^\bullet[n]$ is a bounded below complex of injectives. Then finally we get

$$(\tilde{\eta}')[m] \circ \tilde{\xi} : \mathcal{F} \to \mathcal{I}^\bullet[n+m]$$

which defines a class in the target Ext module.

Of course one has to prove that the resulting class is indepedent of the choices we made. The Ext pairing is associative in an obvious manner. The Ext pairing is functorial in all three variables $\mathcal{F}, \mathcal{G}, \mathcal{H}$ when formulated suitably. Finally, the Ext pairing is compatible with short exact sequences and the boundary maps associated to them (again formulated suitably).

Special case: if $\mathcal{F} = \mathcal{O}_X$ we obtain

$$\mathrm{Ext}^n_X(\mathcal{G}, \mathcal{H}) \times H^m(X, \mathcal{G}) \longrightarrow H^{n+m}(X, \mathcal{H})$$

## 4. Serre duality

Let $X/k$ be proper as above and let $(\omega_X, t)$ be a dualizing sheaf. Using the Ext pairing we obtain pairings

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \times H^{\dim X - i}(X, \mathcal{F}) \longrightarrow k, \quad (\eta, \xi) \longmapsto t(\eta \circ \xi)$$

for any coherent $\mathcal{O}_X$-module $\mathcal{F}$.

**Theorem 4.1.** *If $X$ is projective, equidimensional, and Cohen-Macaulay, then these pairings are perfect.*

More is true, see Hartshorne Theorem 7.6, Chapter III.

*Sketch of proof.* We are going to use $\delta$-functors, see Section 010P. We show that both the (homological) $\delta$-functor

$$\{\mathrm{Ext}^i_X(\mathcal{F}, \omega_X)\}_{i=0,1,2,\ldots}$$

and the (homological) $\delta$-functor

$$\{k\text{-linear dual of } H^{\dim X - i}(X, \mathcal{F})\}_{i=0,1,2,\ldots}$$

are universal. Since for $i = 0$ the functors are isomorphic by our choice of $\omega_X$, we conclude (except that we have to check that the resulting isomorphism comes from the pairings we have discussed in the previous section).

By the (categorical dual of) Lemma 010T it suffices for every $i > 0$ and every coherent module $\mathcal{F}$ to find a surjection $\mathcal{G} \to \mathcal{F}$ such that

$$\mathrm{Ext}^i_X(\mathcal{G}, \omega_X) = 0 \quad \text{and} \quad H^{\dim X - i}(X, \mathcal{G}) = 0$$

Choose a closed immersion $X \to \mathbf{P}^n_k$. By our earlier considerations it suffices to show that

$$\mathrm{Ext}^i_X(\mathcal{O}_X(m), \omega_X) = H^i(X, \omega_X(-m)) \quad \text{and} \quad H^{\dim X - i}(X, \mathcal{O}_X(m)) = 0$$

for $m \ll 0$. The first one holds by our general result about cohomology of twists of coherent sheaves on $\mathbf{P}^n_k$. For the second one we have to use the equidimensionality and Cohen-Macaulay properties of $X$ as it isn't true for a general projective $X$ over $k$. We will do this on the next page. $\square$

## 5. Vanishing and Cohen-Macaulay

We aim to prove the following.

**Proposition 5.1.** *Let $X \subset \mathbf{P}^n_k$ be a closed subscheme equidimensional of dimension $d$, and Cohen-Macaulay. Then for $j < d$ we have $H^j(X, \mathcal{O}_X(m)) = 0$ for $m \ll 0$.*

We proceed in a manner different from what is done in Hartshorne.

**Lemma 5.2.** *For $X$ as in the proposition any finite morphism $X \to \mathbf{P}^d_k$ is finite locally free.*

*Explanation and proof.* A morphism $f : X \to Y$ is said to be *finite locally free* if $f$ is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. If $Y$ is Noetherian, it is equivalent to say that $f$ is finite and flat, see Lemma 02KB.

The lemma holds by "miracle flatness", more precisely Lemma 00R4. Let us check that the lemma applies. Namely, if $x \in X$ is a closed point mapping to $y \in Y = \mathbf{P}^d_k$ then we have $\dim(\mathcal{O}_{X,x}) = d$ and $\dim(\mathcal{O}_{Y,y}) = d$ by dimension theory, the local ring $\mathcal{O}_{Y,y}$ is regular (by our work previous semester), and the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay by assumption. Finally, the quotient $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ has dimension 0 as its spectrum has only the point $x$ in it due to the fact that we assumed $X \to Y$ to be finite. □

**Lemma 5.3.** *For any projective scheme $X$ of dimension $d$ over an infinite field $k$ there exists a finite morphism $f : X \to \mathbf{P}^d_k$ such that $f^*\mathcal{O}_{\mathbf{P}^d_k}(1) = \mathcal{O}_X(1)$.*

*Proof.* Use projections, see next page. □

*Proof of proposition.* There is a reduction to the case where the ground field $k$ is infinite (flat base change + field extensions are flat). Then we may assume we have a finite and flat morphism $f : X \to \mathbf{P}^d_k$ with $f^*\mathcal{O}_{\mathbf{P}^d_k}(1) = \mathcal{O}_X(1)$. Then we see that

$$H^j(X, \mathcal{O}_X(m)) = H^j(\mathbf{P}^d_k, f_*(\mathcal{O}_X(m))) = H^j(\mathbf{P}^d_k, f_*(\mathcal{O}_X)(m))$$

the last equality by the projection formula, see Lemma 01E8. Since $\mathcal{E} = f_*(\mathcal{O}_X)$ is a finite locally free module on $\mathbf{P}^d_k$ we see that it suffices to show the next lemma. □

**Lemma 5.4.** *For any finite locally free module $\mathcal{E}$ on $\mathbf{P}^d_k$ we have $H^j(\mathbf{P}^d_k, \mathcal{E}(m)) = 0$ for $m \ll 0$.*

*Proof.* By duality on $\mathbf{P}^d_k$ this translates into $H^{d-j}(\mathbf{P}^d_k, \mathcal{E}^\vee \otimes \omega_X(-m))$ which does indeed vanish for $-m \gg 0$. □

## 6. Projections

Let $x \in \mathbf{P}^n_k$ be a $k$-rational point. Projection from $x$ is a morphism of varieties

$$p : \mathbf{P}^n_k \setminus \{x\} \longrightarrow \mathbf{P}^{n-1}_k$$

with the property that

$$p^*(\mathcal{O}_{\mathbf{P}^{n-1}_k}(1)) = \mathcal{O}_{\mathbf{P}^n_k}(1)|_{\mathbf{P}^n_k \setminus \{x\}}$$

The easiest way to define it is to choose linear polynomials $L_0, \ldots, L_{n-1} \in k[T_0, \ldots, T_n]$ whose common vanishing set is exactly $\{p\}$ and to use the morphism into $\mathbf{P}^{n-1}_k$ corresponding to the line bundle $\mathcal{O}_{\mathbf{P}^n_k}(1)|_{\mathbf{P}^n_k \setminus \{x\}}$ on $\mathbf{P}^n_k \setminus \{x\}$ and the sections $L_0, \ldots, L_{n-1}$ which generate it (so we get our projection morphism by the universal property of the projective space of dimension $n-1$).

If $x = [0 : \ldots : 0 : 1]$ then it is customary to choose $T_0, \ldots, T_{n-1}$ and the morphism $p$ is simply given on points by

$$p([a_0 : \ldots : a_n]) = [a_0 : \ldots : a_{n-1}]$$

The $k$-rational points of the fibre of $p$ over the $k$-rational point $[b_0 : \ldots : b_{n-1}]$ is the set of points

$$\{[b_0 : \ldots : b_{n-1} : b], b \in k\} \cup \{x\}$$

which is exactly the line connecting $x$ to $[b_0 : \ldots : b_{n-1} : 0]$. In general the fibres of $p$ are exactly the lines passing through $x$.

**Proof of Lemma 5.3.** Let $X \subset \mathbf{P}^n_k$ be a closed subscheme of dimension $d$. Then if $x \in \mathbf{P}^n_k$ is a $k$-rational point not contained in $X$ (and there always is such a point if $d < n$ and $k$ is infinite) then we can consider the restriction

$$p|_X : X \longrightarrow \mathbf{P}^{n-1}_k$$

Observe that the fibres of $p|_X$ are finite because none of the lines passing through $x$ can be completely contained in $X$ as $x \notin X$! Thus $p|_X$ is a finite morphism of schemes, see for example Lemma 02OG. Denote $X' \subset \mathbf{P}^{n-1}_k$ the scheme theoretic image of this morphism (this is the smallest closed subscheme through which $p|_X$ factors, see Section 01R5). We obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d] & X' \arrow[d] \\ \mathbf{P}^n_k \setminus \{x\} \arrow[r, "p"] & \mathbf{P}^{n-1}_k \end{tikzcd}$$

Note that $g$ is finite because $p|_X$ is finite. By induction on $n$ we can find a composition of projections which determines a finite morphism

$$f' : X' \longrightarrow \mathbf{P}^d_k$$

Then $f = f' \circ g : X \to \mathbf{P}^d_k$ is finite as a composition of finite morphisms and the proof is complete.