Problem set 9 for Representations of Finite Groups

Notation. Let G be a finite group. In this problem set we denote G' be the commutator subgroup of G, i.e., the subgroup of G generated by all commutators; equivalently, G' is a normal subgroup of G and $G/G' = G^{ab}$ is the maximal abelian quotient of G.

Exercise 1. This is a converse to Exercise 4 of the last problem set. Let G be a finite group with exactly $|G^{ab}| + 1$ conjugacy classes. Show that G has exactly 1 irreducible representation of degree > 1.

Exercise 2. This is a continuation of Exercise 4 of the last problem set. So, let G be a finite group which has exactly 1 irreducible representation whose degree (i.e., dimension) is > 1. We already know that

(a) G' is not the trivial group,
(b) the number of conjugacy classes of G is equal to $|G^{ab}| + 1$.
(c) if $x, y \in G'$ are two elements with $x \neq 1$ and $y \neq 1$ then x and y are conjugate in the group G,
(d) there exists a prime number p such that for all $x \in G'$ we have $x^p = 1$.

Continue this by the following steps.

(1) Show that G' is a p-group (just look up the definition and it'll be obvious from what we already know).
(2) Show that G' is an elementary abelian p-group, i.e., that G is isomorphic to C_p^a for some integer $a \geq 1$. Hint: use (c) and the fact that any nontrivial p-group has a nontrivial central element (you may use this).
(3) Give an example of such a group.

Exercise 3. Let G be a nonabelian finite group which is not one of the groups studied in Exercise 2. Show there exist distinct irreducible representations (V_i, π_i), $i = 1, 2$ such that $(V_1 \otimes V_2, \pi_1 \otimes \pi_2)$ is not irreducible.

Exercise 4. Let $n \geq 1$. Let V be an n-dimensional complex vector space. Consider the vector space $W = V \otimes V \otimes V$. Recall that W is generated by the "pure" tensors $v_1 \otimes v_2 \otimes v_3$ for $v_1, v_2, v_3 \in V$. There is an action of S_3 on W which is given by the rule

$$\sigma \cdot (v_1 \otimes v_2 \otimes v_3) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}$$

for pure tensors and extended by linearity to general elements of W. Warning: this is not the same as any tensor product representation we have considered prior to this question. Recall that S_3 has 3 irreducible representations, namely $triv, sgn, U$ of dimensions 1, 1, and 2 where $triv$ means the trivial representation. Thus we may write

$$W \cong triv^{\oplus a_n} \oplus \epsilon^{\oplus b_n} \oplus U^{\oplus c_n}$$

as a representation of S_3. Find the integers a_n, b_n, c_n.