Problem set 8 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Show that the 2×2 complex matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}$$

where $\zeta = \exp(2\pi i/3)$ is a primitive third root of 1 do not generate a finite group inside $\mathrm{GL}_2(\mathbf{C})$.

Exercise 2. Consider a finite group G with center C such that the quotient G/C is abelian. Carefully prove there exists a map of sets

$$e : G/C \times G/C \to C$$

with the following property: if $g_1, g_2 \in G$ have images $\overline{g}_1, \overline{g}_2$ in G/C, then $e(\overline{g}_1, \overline{g}_2) = g_1 g_2 g_1^{-1} g_2^{-1}$. Furthermore, show that

(1) e is multiplicative in the first variable, i.e., $e(x_1 x_2, y) = e(x_1, y)e(x_2, y)$
(2) e is anti-symmetric, i.e., $e(x, y) = e(y, x)^{-1}$.

Exercise 3. Let $C \subset G$ be as in Exercise 2. Let p be a prime number and assume that G can be generated by elements g with $g^p = 1$.

(1) If p is odd, show that $g^p = 1$ for all $p \in G$. Hint: if $g_1, g_2 \in G$ are two elements of order p then "compute" $(g_1 g_2)^p$.
(2) If $p = 2$, give an example where this fails, i.e., where G has an element which is not of order 1 or 2.

Exercise 4. Let G be a finite group which has exactly 1 irreducible representation whose degree (i.e., dimension) is > 1. Let G' be the commutator subgroup of G; equivalently, G' is a normal subgroup of G and $G/G' = G^{ab}$ is the maximal abelian quotient of G.

(1) As a warm up, show that G' is not the trivial group, i.e., has more than one element.
(2) Show that the number of conjugacy classes of G is equal to $|G^{ab}| + 1$.
(3) Show that if $x, y \in G'$ are two elements with $x \neq 1$ and $y \neq 1$ then x and y are conjugate in the group G, in other words, show that G' is the union of two conjugacy classes in G.
(4) Conclude that there exists a prime number p such that for all $x \in G'$ we have $x^p = 1$.