Problem set 7 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Let V be a finite dimensional $\mathbf{C}$-vector space. Let

$$\pi_t : G \longrightarrow GL(V), \quad t \in [0,1]$$

be a continuous family of representations of G. This means that for each $t \in [0,1]$ the map π_t is a homomorphism and that for each fixed $g \in G$ the function $[0,1] \to GL(V)$, $t \mapsto \pi_t(g)$ is continuous[1]. Prove that π_0 and π_1 are isomorphic representations of G.

Exercise 2. Let p be an odd prime number. Let G be the group

$$G = \langle a, b, c \rangle / relations$$

where the relations are

(a) the elements a, b, c have order p,
(b) c is central,
(c) $aba^{-1}b^{-1} = c$.

You may use that in this group every element has a unique expression of the form $a^i b^j c^k$ with $0 \leq i, j, k < p$.

(1) describe the conjugacy classes in G (hint: see what happens when you conjugate $a^i b^j c^k$ by a, b, c),
(2) compute the degree 1 characters of G,
(3) describe the remaining irreducible representations of G, for example give their dimensions and the number of them (hint: consider the normal abelian subgroup H of G generated by a and c and use induction plus Mackey's criterion).

[1] Here $[0,1] \subset \mathbf{R}$ is the closed interval. Choosing a basis of V we may think of π_t as a homomorphism into $GL_n(\mathbf{C})$ where $n = \dim(V)$, so the condition is that the map $[0,1] \to GL_n(\mathbf{C})$, $t \mapsto \pi_t(g)$ is continuous, which in turn means that the matrix coefficients $\pi_t(g)_{ij}$ are continuous maps $[0,1] \to \mathbf{C}$.