Problem set 6 for Representations of Finite Groups

If you find errors in this text, please email me, thanks!

Exercise 1. Let G be a finite group. Prove the following are equivalent

(1) G is a simple group, and
(2) for every non-trivial irreducible character χ of G and all $g \in G$, $g \neq 1$ we have $\chi(g) \neq \chi(1)$.

Exercise 2. Let $triv, sgn, V$ denote the irreducible representations of S_3 of dimensions $1, 1, 2$ where $triv$ is the trivial representation and sgn is the sign represenation. Consider the diagonal map $H = S_3 \to S_3 \times S_3 = G$ sending σ to (σ, σ).

(1) List the irreducible representations of G. Hint: recall that these are tensor products of irreducible representations of S_3 and S_3, see problem set 3 Ex 1. So this is just a matter of listing them.
(2) For each of these compute what you get when you restrict to H, i.e., say how many copies of $triv, sgn, V$ are contained in them. (Hint: All but 1 are easy to see but for some you can use a character computation. Maybe only give the explanation in that case to save time.)
(3) Compute the induction $\mathrm{Ind}_H^G(V)$, i.e., write it as a sum of irreducibles of G. Hint: Use Frobenius reciprocity and your computations in part 2.

Exercise 3. Let G be a finite group. Let (V, π) be a representation of G. We say (V, π) is a *monomial representation* if there exists a basis $e_1, \ldots, e_n$ of V such that for each $g \in G$ the map $\pi(g)$ permutes the e_i up to scalars. In other words, for each $g \in G$ and $1 \leq i \leq n$ there exists a $j \in \{1, \ldots, n\}$ and a $\lambda \in \mathbf{C}^*$ such that $\pi(g)(e_i) = \lambda e_j$. Show that a monomial representation is a direct sum of representations of the form $\mathrm{Ind}_H^G(\chi)$ where $H \subset G$ is a subgroup and $\chi : H \to \mathbf{C}^*$ is a character.