Problem set 5 for Representations of Finite Groups

If you find a mistake, please let me know.

Induced representations. Let G be a finite group. Let $H \subset G$ be a subgroup. Given a finite dimension $\mathbf{C}$-representation (W, θ) of H we know from the lectures there exists a unique (up to isomorphism) induced representation (V, π) of G. We will use the **notation** $(V, \pi) = \mathrm{Ind}_H^G(W, \theta)$ or $V = \mathrm{Ind}_H^G(W)$ or $\pi = \mathrm{Ind}_H^G(\theta)$ to indicate this.

Exercise 1. Consider the subgroup $H = S_5 \subset S_6 = G$. Let $\chi : S_5 \to \{\pm 1\} \subset \mathbf{C}^*$ be the sign character which we also may and do view as a 1-dimensional irreducible representation of H. Let (V, π) be the irreducible 4-dimensional representation of $G = S_5$ which is a summand of the standard permutation representation of S_5 on $\mathbf{C}^5$. In the following, please use that the conjugacy classes of the symmetric groups correspond to cycle types.

(1) Compute the character of the representation $\mathrm{Ind}_H^G(\chi)$.
(2) Compute the character of the representation $\mathrm{Ind}_H^G(\pi)$.
(3) How many irreducible constituents does $\mathrm{Ind}_H^G(\chi)$ have?
(4) How many irreducible constituents does $\mathrm{Ind}_H^G(\pi)$ have?

Exercise 2. Let H be a subgroup of a finite group G. Let W_1, W_2 be nonzero representations of H. Show that there is a nonzero map

$$\mathrm{Ind}_H^G(W_1 \otimes W_2) \longrightarrow \mathrm{Ind}_H^G(W_1) \otimes \mathrm{Ind}_H^G(W_2)$$

of representations of G. Hint: you can try to compute using characters, or you can use Serre's Lemma 1 from Section 3.3 (this is the method I prefer).

Exercise 5. Algebraic Integers.

(1) Why is $1/\sqrt{2}$ not an algebraic integer?
(2) Why is $1 + (1/3)\sqrt{2}$ not an algebraic integer?
(3) Find a monic polynomal f with integer coefficients such that $f(\sqrt{3}+i\sqrt{5}) = 0$ where $i \in \mathbf{C}$ is a square root of -1.

Exercise 3. This exercise will be used in proving Jordan's theorem about finite subgroups of $GL_n(\mathbf{C})$ having "large" commutative subgroups. Fix an integer $n \geq 2$. For a vector $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ we denote $|z| = \sqrt{\sum z_i \bar{z}_i}$ the length of the vector z as usual. Given a complex $n \times n$ matrix $A = (a_{ij})$ we define

$$||A|| = \max_{z \in \mathbf{C}^n,\ z \neq 0} |A(z)|/|z|$$

This is known as the operator norm of the matrix A. It is ≥ 0 and equal to 0 if and only if A is the zero matrix. Below, we denote 1 the $n \times n$ identity matrix and A and B are complex $n \times n$ matrices. You can skip parts (1) – (3) if they are obvious to you.

(1) Briefly explain why $||A + B|| \leq ||A|| + ||B||$.
(2) Briefly explain why $||A + B|| \geq ||A|| - ||B||$.
(3) Briefly explain why $||AB|| \leq ||A|| \cdot ||B||$.
(4) Show that $||1 - BA|| \leq ||1 - A|| + ||1 - B|| + ||1 - A|| \cdot ||1 - B||$.
(5) Show that $||AB|| \geq (1 - ||1 - B||)||A||$.

(6) Prove that for A and B invertible and $||1 - A|| \leq \epsilon_1$ and $||1 - B|| \leq \epsilon_2$ we have

$$||1 - ABA^{-1}B^{-1}|| \leq f(\epsilon_1, \epsilon_2) = \frac{2\epsilon_1\epsilon_2}{1 - \epsilon_1 - \epsilon_2 - \epsilon_1\epsilon_2}$$

provided that $1 - \epsilon_1 - \epsilon_2 - \epsilon_1\epsilon_2 > 0$. Hint: set $U = 1 - ABA^{-1}B^{-1}$ and $V = BA - AB = (1 - B)(1 - A) - (1 - A)(1 - B)$. Then $V = UBA$ and hence by (4) and (5) we can find a lower bound for $||V||$ in terms of $||U||$ and ϵ_1, ϵ_2. But we also have an upper bound for $||V||$ in terms of ϵ_1 and ϵ_2 by (1) and (3).