Problem set 4 for Representations of Finite Groups

Exercise 1. Let G be a finite group and let X be a finite set endowed with a G-action. Consider the permutation representation $\mathbf{C}[X]$. Let $\chi : G \to \mathbf{C}$ be the character of $\mathbf{C}[X]$.

(1) Give a combinatorial formula for $\langle \chi, \chi \rangle$ (so in terms of the action of G on X).
(2) Show that if G acts doubly transitive on X and $|X| > 1$, then $\mathbf{C}[X]$ decomposes as the sum of exactly 2 irreducible representations.
(3) What are the dimensions of the two irreducible representations you found in part 2?
(4) Give an example of a transitive action of a G on a set X with $|X| > 1$ such that $\mathbf{C}[X]$ decomposes into more than 2 irreducibles.

Exercise 2. Let p be a number. Let $k = \mathbf{F}_p$ be the field with p elements. Let $G = GL_2(k)$.

(1) Denote $\mathbf{P}^1(k)$ the set of nonzero vectors in $k^2 = k \oplus k$ up to scaling. So $\mathbf{P}^1(k) = (k^2 \setminus \{(0,0)\})/k^*$. Show that $\mathbf{P}^1(k)$ has $p+1$ elements.
(2) Show that there is a natural action of G on $\mathbf{P}^1(k)$. (Give the definition of the action and briefly explain.)
(3) Show this action is doubly transitive. Hint: change of basis matrix.
(4) Conclude, using Exercise 1, that G has at least one irreducible representation of dimension p.

Exercise 3. Let G be a finite group which is equal to its own commutator subgroup (the subgroup generated by commutators). Show that every homomorphism $\pi : G \to GL_n(\mathbf{C})$ maps into $SL_n(\mathbf{C})$.